

## EQUIVARIANT INTERSECTION FORMS, KNOTS IN $S^4$ , AND ROTATIONS IN 2-SPHERES

STEVEN P. PLOTNICK<sup>1</sup>

**ABSTRACT.** We consider the problem of distinguishing the homotopy types of certain pairs of nonsimply-connected four-manifolds, which have identical three-skeleta and intersection pairings, by the equivariant isometry classes of the intersection pairings on their universal covers. As applications of our calculations, we: (i) construct distinct homology four-spheres with the same three-skeleta, (ii) generalize a theorem of Gordon to show that any nontrivial fibered knot in  $S^4$  with odd order monodromy is not determined by its complement, and (iii) give a more constructive proof of a theorem of Hendriks concerning rotations in two-spheres embedded in three-manifolds.

**0. Introduction.** In this paper we are interested in several sorts of “twists” on low-dimensional manifolds, and their relationships. We consider three situations:

(i) In constructing four-manifolds by performing surgery on simple loops, one makes a framing choice. The two possible choices are related by the twist coming from  $\pi_1(SO(3)) \cong \mathbb{Z}_2$ . How does the choice affect the homotopy type of the resulting manifold?

(ii) At most two knots in  $S^4$  have the same complement, the possible difference given by a twist in gluing a regular neighborhood of the knot to its exterior. How does the choice affect the knot type (a relative version of (i))?

(iii) If a three-manifold  $M_0^3$  has  $\partial M_0 = S^2$ , one may define the rotation  $\rho$  in  $S^2$  (see §1). Is  $\rho \cong \text{id}_{M_0} \text{ (rel } \partial M_0)$ ?

It turns out that by considering these situations from the point of view of intersection forms on four-manifolds, we can give fairly complete answers (for certain cases of (i) and (ii)). For instance, we prove

**COROLLARY 3.5.** *There exist (infinitely many pairs of) homology 4-spheres which have the same 3-skeleton but distinct homotopy type.*

Concerning (ii), we have the following generalization of a result of C. McA. Gordon [8, Proposition 4.2]:

**THEOREM 6.2.** *Let  $K$  be any nontrivial fibered knot in  $S^4$  with odd order monodromy. Then  $K$  is not determined by its complement.*

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As for (iii), we have a complete answer, giving a new proof of a result of H. Hendriks [10] concerning certain homotopy equivalences of three-manifolds:

**THEOREM 7.4.** *Let  $M$  be a closed 3-manifold, and let  $M_0 = M - \mathring{B}^3$ . Then the rotation  $\rho$  in  $\partial M_0$  is homotopic to  $\text{id}_{M_0} \text{ (rel } \partial M_0)$  if and only if every summand of  $M$  is either  $S^2 \times S^1$ ,  $S^2 \tilde{\times} S^1$ ,  $\cong P^2 \times S^1$ , or  $\Sigma/\pi$ , where  $\Sigma$  is a homotopy 3-sphere,  $\pi$  is a finite group acting freely on  $\Sigma$ , and all Sylow subgroups of  $\pi$  are cyclic.*

Actually, for the only if direction, we only consider the case where  $M$  has no two-sided projective planes, since our main interest is the case  $M = \Sigma/\pi$ . Our proof relies on some of Hendriks' work, but has a more constructive flavor. It turns out that the secondary obstructions which arise as we try to homotop  $\rho$  to  $\text{id}_{M_0} \text{ (rel } \partial M_0)$  are closely related to equivariant isometries of intersection pairings on certain "spun" 4-manifolds. By studying these isometries we can *completely* understand how the homotopy behaves (when it exists) on the 1-cells of  $M_0$ .

**1. Definitions, notation, results.** Since the common thread running through these results is intersection forms on 4-manifolds, we recall this definition. Let  $M^4$  be a closed, orientable 4-manifold. Define a symmetric, bilinear pairing by

$$H^2(M; \mathbf{Z})/\text{Torsion} \otimes H^2(M; \mathbf{Z})/\text{Torsion} \rightarrow \mathbf{Z}, \quad a \otimes b \rightarrow \langle a \cup b, [M] \rangle.$$

Poincaré duality shows that this form is nonsingular. Alternatively, duality gives an equivalent form

$$H_2(M; \mathbf{Z})/\text{Torsion} \otimes H_2(M; \mathbf{Z})/\text{Torsion} \rightarrow \mathbf{Z}, \quad \alpha \otimes \beta \rightarrow \alpha \cdot \beta,$$

where  $\alpha \cdot \beta$  denotes geometric intersection number.

If  $M^4$  is simply connected, Whitehead showed that the intersection form determines the oriented homotopy type of  $M$  [21]. If  $\pi_1 M \neq \{1\}$ , pass to the universal cover  $\tilde{M}$ . Letting  $[\tilde{M}]$  denote the fundamental class of  $\tilde{M}$ , with possibly infinite chains, we have

$$\begin{array}{ccccccc} \pi_2(M) \otimes \pi_2(M) & \cong & H_2(M; \mathbf{Z}\pi_1 M) & \otimes & H_2(M; \mathbf{Z}\pi_1 M) & & \\ & & \parallel & & & & \\ & & H^2(M; \mathbf{Z}\pi_1 M) & \otimes & H^2(M; \mathbf{Z}\pi_1 M) & \xrightarrow{\cup} & H^4(M; \mathbf{Z}\pi_1 M) \\ & & & & & & \downarrow \cap [\tilde{M}] \\ & & & & & & H_0(M; \mathbf{Z}\pi_1 M) \cong \mathbf{Z}', \end{array}$$

where  $\mathbf{Z}'$  denotes  $\mathbf{Z}$  twisted by the first Stiefel-Whitney class of  $M$ . Via covering transformations,  $\pi_1(M)$  acts as isometries of the form. Assuming  $M$  is oriented, this form is compatible with the intersection form on  $H_2(M)$  in the following sense: Let  $\rho: \pi_2(M) \rightarrow H_2(M)$  be the Hurewicz map. Then the form on  $H_2$ , restricted to the subgroup of spherical cycles,  $\rho(\pi_2)$ , is given by  $\rho(x) \cdot \rho(y) = \sum_{g \in \pi_1} x \cdot gy$ , where only finitely many of these terms are nonzero.

A homotopy equivalence  $f: M \rightarrow N$  lifts to a proper homotopy equivalence  $\tilde{f}: \tilde{M} \rightarrow \tilde{N}$  inducing an equivariant isometry of the forms. If  $M$  and  $N$  are oriented, this is compatible with the isometry induced on  $H_2$ , as described above. We are thus led to the following homotopy invariants:

- (i)  $\pi_1(M)$ ,
- (ii)  $\pi_2(M)$  as a  $\mathbf{Z}\pi_1(M)$ -module,

(iii)  $k \in H^3(\pi_1(M); \pi_2(M))$ , the first  $k$ -invariant of Eilenberg-Mac Lane-Whitehead [17], and

(iv) The equivariant isometry class of  $\pi_2 \otimes \pi_2 \rightarrow \mathbf{Z}$ , together with  $H_2 \otimes H_2 \rightarrow \mathbf{Z}$ , related as above.

Now, (i)–(iii) are carried by the 3-skeleton of  $M$ , but (iv) depends strongly on the attaching map for the top 4-cell of  $M$ . For example,  $S^2 \times S^2$  and  $S^2 \tilde{\times} S^2$ , the nontrivial  $S^2$  bundle over  $S^2$ , differ only in the attaching map for the top cell, and this is reflected in the nonisometric intersection forms  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ .

This list is by no means a complete set of homotopy invariants for closed four-manifolds. For example, let  $L(n, m)$  and  $L(n, m')$  be nonhomotopy equivalent 3-dimensional lens spaces. Then  $L(n, m) \times S^1 \neq L(n, m') \times S^1$ , as can be seen by passing to infinite cyclic covers, although  $\pi_2 = 0$ . The obstruction to a homotopy equivalence lies in the third homology group of the universal cover. Since

$$\begin{aligned} (\# \text{ ends of } \mathbf{Z}_n \times \mathbf{Z} - 1) &= \text{rank } H_c^1(\widetilde{L(n, m) \times S^1}; \mathbf{Z}) \\ &= \text{rank } H_3(S^3 \times \mathbf{R}; \mathbf{Z}) = 1, \end{aligned}$$

we might view the obstruction as reflecting the fact that  $\mathbf{Z}_n \times \mathbf{Z}$  has two ends. For groups with one end, however, it seems possible that (i)–(iv) above are a complete list of homotopy invariants.

This naturally raises the question of whether there are examples of closed four-manifolds, with one end, distinguished by intersection forms on  $\pi_2$ . We give such examples in §3.

We now describe our results in more detail, and establish notation. First, recall a familiar diffeomorphism of  $S^2 \times S^1$ . Let the *twist*  $\tau: S^2 \times S^1 \rightarrow S^2 \times S^1$  be given by

$$\tau(x, \theta) = (\rho_\theta(x), \theta),$$

where  $\rho_\theta$  is a polar rotation of  $S^2$  through angle  $\theta$ . It is well known that  $\tau$  generates  $\pi_1(SO(3)) \cong \mathbf{Z}_2$ ,  $\tau$  does not extend over  $S^2 \times D^2$ , and, up to orientation reversals, is the only nontrivial homotopy equivalence of  $S^2 \times S^1$  [7].

Now let  $M^3$  be a closed 3-manifold, with basepoint  $*$ . If we form  $M \times S^1$ , and perform surgery on the curve  $* \times S^1$  with either of two possible framings (corresponding to  $\pi_1(SO(3)) = \mathbf{Z}_2$ ), we obtain a four-manifold with the same fundamental group as  $M$ . We call either of these a *spin* of  $M$ . More precisely, let  $M_0 \equiv M - \mathring{B}^3$  be a *punctured* copy of  $M$ . Via the product structure of  $M_0 \times S^1$ , there is a natural identification of  $\partial(M_0 \times S^1)$  with  $S^2 \times S^1$ . Let

$$s(M) \equiv M_0 \times S^1 \cup_{\text{id}} S^2 \times D^2$$

be the untwisted spin of  $M$ , and let

$$s'(M) \equiv M_0 \times S^1 \cup_{\tau} S^2 \times D^2$$

be the twisted spin of  $M$ . It is easy to see that  $s(M)$  and  $s'(M)$  have the same 3-skeleton, but different attaching maps for the top cell. Note also that the spins contain natural 2-spheres  $S^2 \times \{0\}$ .

We begin analyzing the spins of  $M$  in §2, where we describe  $\pi_2$ . We also examine some special cases. In particular, the case when  $M$  is a lens space provides the motivation for much of this work, especially §§4, 6, and 7.

In §3 we consider the case when  $M$  is *aspherical*, i.e.  $\tilde{M}$  is contractible. Our results are strongest in this case:

**THEOREM 3.1.** *Let  $M^3$  be a closed, aspherical 3-manifold. There is no  $\pi_1$ -equivariant map  $\pi_2(s(M)) \rightarrow \pi_2(s'(M))$  preserving the intersection forms. Consequently,  $s(M) \neq s'(M)$ .*

Corollary 3.5 follows directly from Theorem 3.1.

The case when  $M$  is a closed, *spherical* 3-manifold, i.e.  $\tilde{M}$  is a homotopy 3-sphere, is more difficult and is treated in §4. First of all, we do not expect Theorem 3.1 to hold here, since  $s(M) \cong s'(M)$  if  $M$  is a lens space. Secondly, the analysis of equivariant isometries is more subtle. Accordingly, we introduce the notion of a *special isometry*, which should be thought of as the algebraic analogue of a homotopy equivalence of pairs  $(s(M), S^2 \times \{0\}) \rightarrow (s'(M), S^2 \times \{0\})$ . The algebraic problem encountered here can actually be solved for any finite group (Theorem 4.4). For finite groups arising from 3-manifolds, our result is given by

**COROLLARY 4.8.** *Suppose  $\pi$  acts freely on a homotopy 3-sphere  $\Sigma$ . Then there is a special isometry  $\pi_2(s(\Sigma/\pi)) \rightarrow \pi_2(s'(\Sigma/\pi))$  if, and only if, all Sylow subgroups of  $\pi$  are cyclic.*

Surprisingly, this class of 3-manifolds includes not just lens spaces, but also the prism manifolds  $(2, 2, \alpha)$ ,  $\alpha$  odd. The fact that special isometries exist for lens spaces is fairly obvious once one knows that lens spaces admit circle actions with fixed points (§2), but prism manifolds do not admit such actions. Nevertheless, this is consistent with, and anticipates, Hendriks' results (§7).

It is now a fairly simple matter to extend these results to an arbitrary closed 3-manifold with no 2-sided  $P^2$ 's. We do this in §5, where we show (Theorem 5.1) that a special isometry exists if and only if every summand of  $M$  is either  $S^2 \times S^1$ ,  $S^2 \tilde{\times} S^1$ , or a spherical manifold as in Corollary 4.8.

The main application of this work, and the original motivation, is in §6, where we discuss the question of whether knots in  $S^4$  are determined by their complements. We first recall the issues involved here. For simplicity, we restrict our attention to knots in  $S^4$ , although everything we say holds equally well in  $S^n$ ,  $n \geq 4$  [1, 7, 13, 14].

Given  $K = (S^4, S^2)$  a smooth knot, let  $X(K) = S^4 \setminus S^2 \times \mathring{D}^2$  be the *exterior* of  $K$ . If  $\psi$  is a diffeomorphism of  $S^2 \times S^1$ , consider the homotopy sphere  $X(K) \cup_{\psi} S^2 \times D^2$ . This construction depends only on the pseudo-isotopy class of  $\psi$ . The pseudo-isotopy classes of  $S^2 \times S^1$  are  $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ , where the first two  $\mathbf{Z}_2$  factors are represented by orientation reversals of  $S^2$  and  $S^1$  respectively, and the last  $\mathbf{Z}_2$  factor is represented by the twist  $\tau$ . Since the orientation reversals extend over  $S^2 \times D^2$ , only  $\tau$  can possibly create a new knot. Thus, there are at most two knots

with the same complement, namely

$$K = (X(K) \cup_{\text{id}} S^2 \times D^2, S^2 \times \{0\})$$

and

$$K^* = (X(K) \cup_{\tau} S^2 \times D^2, S^2 \times \{0\}).$$

Furthermore,  $K$  and  $K^*$  are equivalent knots if and only if  $\varepsilon\tau$  extends over  $X(K)$ , where  $\varepsilon$  represents an element of the first two  $\mathbf{Z}_2$  factors.

Examples where  $K \neq K^*$  have been given by Cappell and Shaneson [2] in dimensions 4, 5, and 6, and by Gordon [8] in dimension 4. Our examples generalize those of Gordon. Very briefly, if  $K$  is fibered with odd order monodromy, then by lifting to the cyclic cover determined by the order, we find that the equivalence of  $K$  and  $K^*$  implies the existence of special isometries. Using Theorem 5.1 and some easily proved restrictions on the possible fibers, we prove Theorem 6.2.

Finally, in §7 we examine a closely related problem concerning homotopy equivalences of three-manifolds. Suppose  $M_0^3$  has  $\partial M_0 = S^2$ , with collar  $S^2 \times [0, 2\pi]$ . Define the *rotation*  $\rho$  in  $S^2$  by

$$\begin{aligned} \rho(y) &= y, & y &\in M_0 \setminus S^2 \times [0, 2\pi], \\ \rho(x, \theta) &= (\rho_\theta(x), \theta), & (x, \theta) &\in S^2 \times [0, 2\pi]. \end{aligned}$$

Notice that  $\rho|_{\partial M_0} = \text{id}|_{\partial M_0}$ , and that  $\rho \simeq \text{id}_{M_0}$  if  $\partial M_0$  is permitted to move during the homotopy. We are interested in whether  $\rho \simeq \text{id}_{M_0} \text{ (rel } \partial M_0)$ .

Following Hendriks, we examine the obstructions which arise as we try to homotop  $\text{id}_{M_0}$  to  $\rho$  (rel  $\partial$ ). These turn out to be closely related to the special isometries of Corollary 4.8. Our knowledge of special isometries leads to an alternate proof of Hendriks' theorem (Theorem 7.4), and also to a complete description of the homotopies on the 1-skeleton of  $M_0$  (Theorem 7.6).

**2. Spinning 3-manifolds.** We are interested in the homotopy types of the spun manifolds  $s(M)$  and  $s'(M)$ . First, here are some special cases which motivate the work of the next few sections.

*Case 1:*  $M = S^3$ . It is well known that  $s(S^3) \cong s'(S^3) \cong S^4$ , since the twist  $\tau$  on  $\partial(B^3 \times S^1)$  extends to a diffeomorphism of  $B^3 \times S^1$ .

*Case 2:*  $M = L^3(n, m)$ . Any lens space  $L^3$  admits an  $S^1$  action with a circle of fixed points [26]. If  $\ast \in \text{Fix}(S^1, L)$ , then the action of  $\theta \in S^1$  on a small invariant transverse disk through  $\ast$  is rotation by  $\theta$ . The action restricts to  $L_0$ , and defines a diffeomorphism  $f: L_0 \times S^1 \rightarrow L_0 \times S^1$  by  $f(x, \theta) = (\theta \cdot x, \theta)$ . Since  $f|_{\partial(L_0 \times S^1)} = \tau$ , we again have  $s(L) \cong s'(L)$ .

Lift this analysis to universal covers. We can describe  $\widetilde{s(L)} \text{ } \widetilde{s'(L)}$  as the result of performing equivariant surgery on the  $n = |\pi|$  lifts of  $\ast \times S^1$  in  $S^3 \times S^1$ , with the untwisted (twisted) framing. If we use the untwisted framing, then the first surgery results in  $S^4$ , and each successive surgery adds a copy of  $S^2 \times S^2$ , so that  $\widetilde{s(L)} \cong \#_1^{n-1} S^2 \times S^2$ . On the other hand, we can lift the  $S^1$  action on  $L$  to an action on  $S^3$ . The fixed circle in  $L$  lifts to a fixed circle in  $S^3$  which passes through

all lifts of  $*$ . Consequently, the diffeomorphism  $\tilde{f}: S^3 \times S^1 \rightarrow S^3 \times S^1$ ,  $\tilde{f}(x, \theta) = (\theta \cdot x, \theta)$ , *simultaneously* changes all the framings on the lifts of  $* \times S^1$ , and we obtain  $s'(L) \cong \#_1^{n-1} S^2 \times S^2$ .

*Case 3:* If  $M = S^2 \times S^1$ ,  $S^2 \tilde{\times} S^1$ , or  $P^2 \times S^1$ , we again have  $s(M) \cong s'(M)$ , since these manifolds also admit circle actions with fixed points [26].

*Case 4:*  $M = S^3/\pi$ , where  $\pi$  is a finite group of order  $n$  acting freely on  $S^3$ . If  $S^3/\pi$  is not a lens space, an  $S^1$  action on  $S^3/\pi$  has no fixed points [26], so the method of Case 2 fails. On the other hand, it does work for the universal covers: As in Case 2, we have  $s(S^3/\pi) \cong \#_1^{n-1} S^2 \times S^2$ . Now pass an unknotted circle in  $S^3$  through all lifts of  $*$ . Since  $S^1$  acts as rotations about this circle, we have that  $s'(S^3/\pi) \cong \#_1^{n-1} S^2 \times S^2$ . The universal covers are diffeomorphic but the obvious diffeomorphism is not the lift of a diffeomorphism, i.e., not equivariant.

*Case 5:*  $M$  aspherical. Assume that  $\tilde{M} = \mathbf{R}^3$  (no counterexamples are known). Pass an unknotted, properly embedded line through the lifts of  $*$ , and let  $S^1$  act as rotations about this line. Then

$$\overline{s(M)} \cong \overline{s'(M)} \cong S^2 \times \mathbf{R}^2 \# \left( \#_1^\infty S^2 \times S^2 \right).$$

Again, since  $M$  does not admit an  $S^1$ -action with fixed points, the diffeomorphism is not equivariant.

As the examples indicate, the spun manifolds have quite a bit in common. To begin analyzing them, we describe  $\pi_2$ . From now on we write  $\pi = \pi_1(M)$ . Recall that the *augmentation ideal*  $I\pi$  is defined as  $I\pi = \ker(\mathbf{Z}\pi \xrightarrow{\epsilon} \mathbf{Z})$ .

**PROPOSITION 2.1.** *Let  $N^4$  denote either spin of  $M^3$ . Then  $\pi_2(N) \cong I\pi \oplus \pi_2(M_0)$  as  $\mathbf{Z}\pi$ -modules.*

**PROOF.** Covering space theory shows that  $\tilde{N} = \tilde{M}_0 \times S^1 \cup (\mathbb{1}_\pi S^2 \times D^2)$ . The Mayer-Vietoris sequence for this decomposition reduces to

$$0 \rightarrow H_2(\tilde{M}_0 \times S^1) \rightarrow H_2(\tilde{N}) \rightarrow \bigoplus_{\pi} H_1(S^2 \times S^1) \rightarrow H_1(\tilde{M}_0 \times S^1) \rightarrow 0,$$

or

$$0 \rightarrow H_2(\tilde{M}_0) \rightarrow H_2(\tilde{N}) \rightarrow \mathbf{Z}\pi \xrightarrow{\epsilon} \mathbf{Z} \rightarrow 0,$$

and  $\ker(\epsilon)$  naturally splits back to  $H_2(\tilde{N})$ . To see this, pick  $*$   $\in \partial M_0$  to be the north pole, and pick a lift of  $*$ , say  $\tilde{*}$ . Given  $g \in \pi$ ,  $g \neq e$ , let  $\gamma_g$  denote a path in  $\tilde{M}_0$  from  $\tilde{*}$  to  $g\tilde{*}$ . Crossing with  $S^1$  gives annuli  $\{\gamma_g \times S^1\}_{g \neq e}$  in  $\tilde{M}_0 \times S^1$ . The boundaries of the annuli are capped off, when we add  $\mathbb{1}_\pi S^2 \times D^2$ , by  $\mathbb{1}_\pi$  (north pole  $\times D^2$ ), to give 2-spheres  $\{S_{g-e}\}_{g \neq e}$ . These spheres are the natural generators (over  $\mathbf{Z}$ ) of  $\ker \epsilon$ .

Note that it is irrelevant here whether  $N$  is  $s(M)$  or  $s'(M)$ , since  $\tau$  restricts to the identity on  $* \times D^2$ . This completes the proof.  $\square$

We might also add that the framing is irrelevant when we consider intersection forms on  $H_2(s(M)) = H_2(s'(M)) \cong H_1(M) \oplus H_2(M)$ , since this just reflects duality on  $M$ . However, the framing *is* relevant when we consider intersection forms on

$\pi_2$ . We have

PROPOSITION 2.2. (i) *The intersection form for  $\pi_2(\widetilde{s(M)})$ , restricted to  $I\pi$ , is the zero form.*

(ii) *The intersection form for  $\pi_2(\widetilde{s'(M)})$ , restricted to  $I\pi$ , is given by  $S_{g-e} \cdot S_{h-e} = 1 + w(g)\delta_{g,h}$ , where  $w = w_1$  is the first Stiefel-Whitney class of  $M$ , and  $\delta_{g,h}$  is the Kronecker delta.*

PROOF. We may assume that the paths  $\gamma_g$  in Proposition 2.1 are disjoint except at  $\bar{*}$ , so that the annuli  $\gamma_g \times S^1$  intersect pairwise in  $\bar{*} \times S^1$ . The spheres  $S_{g-e}$  and  $S_{h-e}$  share a common disk  $\bar{*} \times D^2$ . In  $\widetilde{s(M)}$ , this disk can be pushed off itself, but for  $\widetilde{s'(M)}$ , a single intersection point (+1) is introduced. This is well known, for example, in the case of the twisted  $S^2$ -bundle over  $S^2$ ,  $S^2 \tilde{\times} S^2 = S^2 \times D^2 \cup_{\tau} S^2 \times D^2$ , and the situation here is identical. This proves the proposition if  $g \neq h$ .

To compute  $S_{g-e} \cdot S_{g-e}$ , translate  $\bar{*} \times D^2$  to  $g\bar{*} \times D^2$ . Again, in  $\widetilde{s(M)}$ , this disk pushes off itself, while in  $\widetilde{s'(M)}$  an intersection point is introduced, with sign depending on whether  $g$  preserves or reverses orientation. This completes the proof.

□

Note that, assuming  $M$  is orientable, the intersection form of  $\widetilde{s'(M)}$ , in the natural basis  $\{S_{g-e}\}_{g \neq e}$ , is given by

$$\begin{pmatrix} 2 & 1 & 1 & \cdots \\ 1 & 2 & 1 & \\ 1 & 1 & 2 & \\ & & & \ddots \end{pmatrix}.$$

**3. Spinning aspherical 3-manifolds.** Let  $M^3$  be aspherical, so that  $\tilde{M}$  is contractible. Then  $\pi_2(M_0) \cong \mathbf{Z}\pi$ , naturally represented by the boundary spheres of  $\tilde{M}_0$ . We write  $\{S_g\}_{g \in \pi}$  for this collection of 2-spheres with  $S_e$  being the lift of  $\partial M_0$  containing  $\bar{*}$ , and  $S_g = g(S_e)$ , as oriented manifolds.

Since the  $S_g$  are pairwise disjoint, and any  $S_g$  can be pushed off itself in  $\tilde{M}_0 \times S^1$ , the intersection form is zero on  $\mathbf{Z}\pi$ , for both  $s(M)$  and  $s'(M)$ .

To compute intersections between  $I\pi$  and  $\mathbf{Z}\pi$ , note that each  $S_{g-e}$  intersects  $S_e$  transversely once, at  $\bar{*}$ , and intersects  $S_g$  once, at  $g\bar{*}$ . Taking orientations into account, we find

$$(1) \quad S_{g-e} \cdot S_e = -1, \quad S_{g-e} \cdot S_h = w(g)\delta_{g,h}, \quad h \neq e,$$

for both  $s(M)$  and  $s'(M)$ . This discussion, along with Proposition 2.2, gives the complete intersection forms.

It is fairly easy to see that the forms are isometric if we disregard the  $\mathbf{Z}\pi$  action. Order the elements of  $\pi$ ,  $e = g_0, g_1, g_2, \dots$ . Define  $\phi: I\pi \oplus \mathbf{Z}\pi \rightarrow I\pi \oplus \mathbf{Z}\pi$  to be the identity on  $\mathbf{Z}\pi$ , and, on  $I\pi$ ,  $\phi(S_{g_i-e}) = S_{g_i-e} + \sum_{j=1}^i S_{g_j}$ . If  $M$  is orientable,  $\phi$  is an isometry from  $\pi_2(\widetilde{s'(M)})$  to  $\pi_2(\widetilde{s(M)})$ . We leave the necessary modifications when  $M$  is not orientable to the reader.

This isometry, however, does not respect the  $\mathbf{Z}\pi$ -module structure on  $\pi_2$ . In fact, we have

**THEOREM 3.1.** *Let  $M^3$  be a closed, aspherical 3-manifold, with spins  $s(M)$  and  $s'(M)$ . There is no  $\mathbf{Z}\pi$ -equivariant map  $\phi: \pi_2(s'(M)) \rightarrow \pi_2(s(M))$ , covering an automorphism of  $\pi$ , which preserves the intersection forms. Consequently,  $s(M) \neq s'(M)$ .*

To prove this, we study maps of  $\pi_2$ . This requires the following simple lemmas.

**LEMMA 3.2.** *Suppose  $\pi$  has one end. Then every  $\mathbf{Z}\pi$ -map  $I\pi \rightarrow \mathbf{Z}\pi$  is the restriction of a  $\mathbf{Z}\pi$ -map  $\mathbf{Z}\pi \rightarrow \mathbf{Z}\pi$ .*

**PROOF.** Applying  $\text{Hom}_{\mathbf{Z}\pi}(\cdot, \mathbf{Z}\pi)$  to the short exact sequence  $0 \rightarrow I\pi \rightarrow \mathbf{Z}\pi \rightarrow \mathbf{Z} \rightarrow 0$  yields

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\mathbf{Z}\pi}(\mathbf{Z}\pi, \mathbf{Z}\pi) &\rightarrow \text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi) \rightarrow \text{Ext}_{\mathbf{Z}\pi}^1(\mathbf{Z}, \mathbf{Z}\pi) \\ &\parallel \\ &H^1(\pi; \mathbf{Z}\pi) = 0, \end{aligned}$$

proving the lemma. Here we used  $\text{Hom}_{\mathbf{Z}\pi}(\mathbf{Z}, \mathbf{Z}\pi) = 0$ , since  $\pi$  is infinite. Note that the map  $\mathbf{Z}\pi \rightarrow \mathbf{Z}\pi$  defined by  $\phi(e) = \sum n_g g$  restricts to the map  $I\pi \xrightarrow{\phi} \mathbf{Z}\pi$  given by  $\phi(h - e) = \sum n_g (hg - g)$ .  $\square$

**LEMMA 3.3.** *Suppose  $\pi$  has zero or one end. Then every  $\mathbf{Z}\pi$ -map  $I\pi \rightarrow I\pi$  can be written uniquely as the restriction of a  $\mathbf{Z}\pi$ -map  $\mathbf{Z}\pi \rightarrow I\pi$ , plus a multiple of the identity.*

**PROOF.** Applying  $\text{Hom}_{\mathbf{Z}\pi}(\cdot, I\pi)$  to  $0 \rightarrow I\pi \rightarrow \mathbf{Z}\pi \xrightarrow{\epsilon} \mathbf{Z} \rightarrow 0$  yields:

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\mathbf{Z}\pi}(\mathbf{Z}\pi, I\pi) &\rightarrow \text{Hom}_{\mathbf{Z}\pi}(I\pi, I\pi) \xrightarrow{\delta} \text{Ext}_{\mathbf{Z}\pi}^1(\mathbf{Z}, I\pi) \rightarrow 0 \\ &\parallel \\ H^0(\pi; \mathbf{Z}\pi) &\xrightarrow{\epsilon_*} H^0(\pi; \mathbf{Z}) \rightarrow H^1(\pi; I\pi) \rightarrow H^1(\pi; \mathbf{Z}\pi) \\ &\parallel \qquad \qquad \qquad \parallel \\ &\mathbf{Z} \qquad \qquad \qquad 0 \end{aligned}$$

Here we used  $\text{Hom}_{\mathbf{Z}\pi}(\mathbf{Z}, I\pi) = 0$ , since  $I\pi$  has no fixed elements. Now  $H^0(\pi; \mathbf{Z}\pi) \cong (\mathbf{Z}\pi)^\pi$ . If  $\pi$  is infinite, there are no fixed elements, so  $\text{Ext}_{\mathbf{Z}\pi}^1(\mathbf{Z}, I\pi) \cong \mathbf{Z}$ , which splits back to  $\text{Hom}_{\mathbf{Z}\pi}(I\pi, I\pi)$ , generated by the identity, proving the lemma. But if  $|\pi| = n$ ,  $(\mathbf{Z}\pi)^\pi \cong \mathbf{Z}$ , generated by the norm element  $N = \sum g$ . The map  $\epsilon_*$  is multiplication by  $n$ , so that  $\text{Ext}_{\mathbf{Z}\pi}^1(\mathbf{Z}, I\pi) \cong \mathbf{Z}_n$ . In this case,  $\delta$  (identity) generates  $\mathbf{Z}_n$ , so every element of  $\text{Hom}_{\mathbf{Z}\pi}(I\pi, I\pi)$  can be written uniquely as the restriction of an element of  $\text{Hom}_{\mathbf{Z}\pi}(\mathbf{Z}\pi, I\pi)$ , plus  $m$  times the identity,  $0 \leq m < n$ .

Given  $\phi: \mathbf{Z}\pi \rightarrow I\pi$ ,  $\phi(e) = \sum_{g \neq e} m_g (g - e)$ , the restriction of  $\phi$  to  $I\pi$  is given by  $\phi(h - e) = \sum_{g \neq e} m_g (hg - h - g + e)$ .  $\square$

**LEMMA 3.4.** *Let  $\pi$  be an infinite group, and let  $A, B \subset \pi$  be finite subsets. Then there exists  $g \in \pi$ ,  $g \cdot A \cap B = \emptyset$ .*



PROOF. Pick  $g$  in the complement of the finite set  $B \cdot A^{-1} = \{ba^{-1} : b \in B, a \in A\}$ .  $\square$

PROOF OF THEOREM 3.1. Suppose  $\phi: I\pi \oplus \mathbb{Z}\pi \rightarrow I\pi \oplus \mathbb{Z}\pi$  preserves the intersection forms, which we write as  $(\ , \ )'$  and  $(\ , \ )$ . Assume  $\phi$  covers the identity map on  $\pi$ . Then, for each  $h, k \in \pi, e \neq h \neq k \neq e$ ,

$$1 = (S_{h-e}, S_{k-e})' = (\phi(S_{h-e}), \phi(S_{k-e})).$$

The intersection form on  $\pi_2(s(M))$  is zero on both  $I\pi \oplus I\pi$  and  $\mathbb{Z}\pi \oplus \mathbb{Z}\pi$ . Hence, in the notation from Lemmas 3.2 and 3.3,

$$(2) \quad 1 = \left( \sum_{g \neq e} m_g \{S_{hg-e} - S_{h-e} - S_{g-e}\} + mS_{h-e}, \sum_{\bar{g}} n_{\bar{g}} \{S_{k\bar{g}} - S_{\bar{g}}\} \right) \\ + \left( \sum_{g \neq e} m_g \{S_{kg-e} - S_{k-e} - S_{g-e}\} + mS_{k-e}, \sum_{\bar{g}} n_{\bar{g}} \{S_{h\bar{g}} - S_{\bar{g}}\} \right).$$

Note that the set of  $g, \bar{g}$  with  $m_g, n_{\bar{g}}$  nonzero is finite, and that the contribution of individual terms in (2) is given by (1).

Using Lemma 3.4, pick  $k$  so that

$$k\bar{g} \neq e \neq kg, \quad k\bar{g} \neq g, \quad kg \neq \bar{g}, \quad k \neq \bar{g}.$$

Now pick  $h$  so that

$$h\bar{g} \neq e \neq hg, \quad h\bar{g} \neq kg, \quad h\bar{g} \neq g, \quad h\bar{g} \neq k, \\ hg \neq k\bar{g}, \quad hg \neq \bar{g}, \quad h \neq k\bar{g}, \quad h \neq \bar{g}.$$

With this choice of  $h, k$ , the right-hand side of (2) becomes

$$-2 \left[ n_e \left( \sum m_g - m \right) + \sum_{g, \bar{g}} m_g n_{\bar{g}} - w(g) \delta_{g, \bar{g}} \right],$$

a contradiction.

This proves the theorem when  $\phi$  covers the identity. More generally, suppose  $\phi$  is an  $\alpha$ -map for some  $\alpha \in \text{Aut } \pi$ . We could go back and redo this proof by considering  $\alpha$ -maps from  $I\pi$  to  $I\pi$  and  $\mathbb{Z}\pi$ . It is perhaps more enlightening to do the following: Since  $M$  is aspherical,  $\alpha^{-1}$  can be geometrically realized by a homotopy equivalence  $f: M \rightarrow M$ . We can assume, for a small ball  $B \subset M$ , that  $f^{-1}(B) = B$  and  $f|_B: B \rightarrow B$  is  $\pm \text{id}$  [4]. Then  $f \times \text{id}_{S^1}$  restricts to a homotopy equivalence of  $M_0 \times S^1$ , and induces a homotopy equivalence  $F: s(M) \rightarrow s(M)$ . Lifting to  $\widetilde{s(M)}$ , we obtain an  $\alpha^{-1}$ -equivariant isometry  $\Psi$  of  $\pi_2(s(M))$ . Then  $\Psi \circ \phi$  covers the identity and preserves the intersection forms. This completes the proof of Theorem 3.1.  $\square$

COROLLARY 3.5. *There exist (infinitely many pairs of spun) homology 4-spheres which have the same 3-skeleton but distinct homotopy type.*

PROOF. Pick  $M$  to be an aspherical homology 3-sphere, and apply Theorem 3.1.  $\square$

REMARK. Spun homology spheres arise naturally in the study of  $S^1$ -actions on 4-manifolds. Work of Fintushel [5] and Pao [27], extended from the case  $\pi_1(N^4) = \{1\}$  to  $H_1(N^4; \mathbb{Z}) = 0$  in [28], shows: Let  $(S^1, N^4)$  be a smooth  $S^1$ -action on a homology 4-sphere  $N$ . Then  $N/S^1 \equiv M$  is a homology 3-sphere (or disk), and  $N$  is either  $s(M)$  or  $s'(M)$ . Thus, at most two homology 4-spheres admit  $S^1$ -actions with

quotient  $M$ . If  $M$  is aspherical, Theorem 3.1 shows that the two homology spheres are distinct, completing the *topological* classification in this case.

**4. Spinning spherical 3-manifolds.** Let  $M^3$  be spherical,  $M = \Sigma/\pi$ , where  $\Sigma$  is a homotopy 3-sphere and  $\pi$  is a finite group of order  $n$  acting freely on  $\Sigma^3$ . The Lefschetz fixed point formula shows that  $M$  is orientable.

Let  $N = \sum_{g \in \pi} g$  be the norm element of  $\pi$ . Then  $\pi_2(M_0) \cong \mathbb{Z}\pi/N$ , naturally generated by the boundary spheres of  $\tilde{M}_0$ . Writing  $\{S_g\}_{g \in \pi}$  for the boundary spheres, we have  $\sum S_g = 0$  in  $H_2(\tilde{M}_0)$ . Fix an ordering of  $\pi$ ,  $e = g_0, g_1, \dots, g_{n-1}$ . Then  $I\pi$  is generated over  $\mathbb{Z}$  by  $\{S_{g_i - e}\}_1^{n-1}$ , and  $\mathbb{Z}\pi/N$  is generated over  $\mathbb{Z}$  by  $\{S_{g_i}\}_1^{n-1}$ . We will always use these bases to express the intersection forms.

Proposition 2.2 and the discussion at the beginning of §3, trivially modified, allow us to express the intersection forms in  $(n-1) \times (n-1)$  blocks as follows. For  $\pi_2(s(M))$  we have

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

and for  $\pi_2(s'(M))$  we have

$$\left( \begin{array}{cccccc|c} 2 & 1 & 1 & \dots & \dots & 1 & \\ 1 & 2 & 1 & & & \vdots & \\ 1 & 1 & 2 & & & \vdots & \\ \vdots & & & \ddots & & 1 & \\ \vdots & & & & \ddots & & \\ 1 & \dots & \dots & \dots & 1 & 2 & \\ \hline & & & & I & & 0 \end{array} \right).$$

Recall the discussion for lens spaces, Case 2, §2. Given a lens space  $L$ , with  $\pi_1(L) = \mathbb{Z}_n$ , there is a circle action on  $L$  with a circle of fixed points, and this circle is a generator  $g$  of  $\mathbb{Z}_n$ . The action lifts to an action on  $S^3$ , with fixed axis the lift of  $g$ . Pick a small ball  $B^3 \subset L$ ,  $* \in \partial B^3$ , and a lift  $\tilde{*} \in \partial \tilde{B}^3 \subset S^3$ . Take the paths  $\gamma_{g^j}$  from Proposition 2.1 to lie on the axis, except for small detours along  $\partial(g^j \tilde{B}) = S_{g^j}$ ,  $1 \leq j \leq i$ , as in Figure 1.

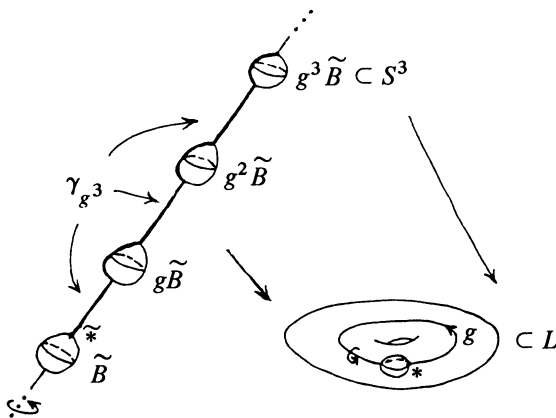


FIGURE 1

The diffeomorphism  $\tilde{f}$  of  $(S^3 \setminus \mathbb{L}, g^i \tilde{B}) \times S^1$ ,  $\tilde{f}(x, \theta) = (\theta \cdot x, \theta)$ , clearly acts by the identity on the spheres  $S_{g^i}$  (the  $\mathbb{Z}\pi/N$  part of  $\pi_2$ ). For  $I\pi$ , note that  $\tilde{f}_*(\gamma_{g^i} \times S^1) = (\gamma_{g^i} \times S^1) + \sum_{j=1}^i S_{g^j}$  in  $H_2((S^3 \setminus \mathbb{L}, g^i \tilde{B}) \times S^1, \partial; \mathbb{Z})$ . Adding  $\mathbb{L} \times S^2 \times D^2$  now gives  $\tilde{f}: \widetilde{s'(L)} \rightarrow \widetilde{s(L)}$ , with  $\tilde{f}_*(S_{g^i}) = S_{g^i}$ ,  $\tilde{f}_*(S_{g^i-e}) = S_{g^i-e} + \sum_{j=1}^i S_{g^j}$ , so  $\tilde{f}_*$  is represented by  $\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$ , where

$$X = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ & 1 & 1 & & 1 \\ & & 1 & & \vdots \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

This determines an isometry between the forms:

$$\begin{pmatrix} I & X' \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} = \left( \begin{array}{cccc|c} 2 & 1 & 1 & \dots & \\ 1 & 2 & 1 & & \\ 1 & 1 & 2 & & \\ \vdots & & & \ddots & \\ & & & & 2 \\ \hline & & & & I \\ & & & & 0 \end{array} \right),$$

since

$$X + X' = \begin{pmatrix} 2 & 1 & \dots \\ 1 & 2 & \\ \vdots & & \ddots \\ & & & 2 \end{pmatrix}.$$

From its geometric origin, this isometry must be equivariant, although the reader might verify directly that  $X$  gives a  $\mathbb{Z}\pi$ -map  $I\pi \rightarrow \mathbb{Z}\pi/N$ .

This suggests we study  $\text{Hom}_{\mathbb{Z}\pi}(I\pi, I\pi)$  and  $\text{Hom}_{\mathbb{Z}\pi}(I\pi, \mathbb{Z}\pi/N)$ . Lemma 3.3 computes  $\text{Hom}_{\mathbb{Z}\pi}(I\pi, I\pi)$ . For  $\text{Hom}_{\mathbb{Z}\pi}(I\pi, \mathbb{Z}\pi/N)$ , we have

**LEMMA 4.1.** *Let the finite group  $\pi$  act freely on a 2-connected space  $\Sigma$ . Then*

$$0 \rightarrow \mathbb{Z}\pi/N \rightarrow \text{Hom}_{\mathbb{Z}\pi}(I\pi, \mathbb{Z}\pi/N) \rightarrow H_1\pi \rightarrow 0$$

*is exact, but does not split.*

**PROOF.** Applying  $\text{Hom}_{\mathbb{Z}\pi}(\_, \mathbb{Z}\pi/N)$  to  $0 \rightarrow I\pi \rightarrow \mathbb{Z}\pi \rightarrow \mathbb{Z} \rightarrow 0$  yields

$$0 \rightarrow \text{Hom}_{\mathbb{Z}\pi}(\mathbb{Z}\pi, \mathbb{Z}\pi/N) \rightarrow \text{Hom}_{\mathbb{Z}\pi}(I\pi, \mathbb{Z}\pi/N) \rightarrow \text{Ext}_{\mathbb{Z}\pi}^1(\mathbb{Z}, \mathbb{Z}\pi/N) \rightarrow 0,$$

since  $\mathbb{Z}\pi/N$  has no fixed elements. Now  $\text{Ext}_{\mathbb{Z}\pi}^1(\mathbb{Z}, \mathbb{Z}\pi/N) \cong H^1(\pi; \mathbb{Z}\pi/N)$ , and the coefficient sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}\pi \rightarrow \mathbb{Z}\pi/N \rightarrow 0$  yields

$$\rightarrow H^1(\pi; \mathbb{Z}\pi) \rightarrow H^1(\pi; \mathbb{Z}\pi/N) \rightarrow H^2(\pi; \mathbb{Z}) \rightarrow H^2(\pi; \mathbb{Z}\pi) \rightarrow .$$

Since  $\pi$  is finite,  $H^1(\pi; \mathbb{Z}\pi) = 0$ , and  $H^2(\pi; \mathbb{Z}\pi) \cong H^2(\Sigma; \mathbb{Z}) = 0$ , since  $\Sigma/\pi$  has the 3-skeleton of a  $K(\pi, 1)$ . Finally,  $H^2(\pi; \mathbb{Z}) \cong H^2(\Sigma/\pi; \mathbb{Z}) \cong H_1(\Sigma/\pi; \mathbb{Z}) \cong H_1(\pi; \mathbb{Z})$ , by universal coefficients. The sequence cannot split as  $\text{Hom}_{\mathbb{Z}\pi}(I\pi, \mathbb{Z}\pi/N)$  is torsion-free.  $\square$

EXAMPLE. Let  $\pi = \mathbf{Z}_n$ . Then

$$\begin{aligned}\mathbf{Z}\pi &= \mathbf{Z}[x]/(x^n - 1), \\ \mathbf{Z}\pi/N &= \mathbf{Z}[x]/(x^n - 1, 1 + x + \cdots + x^{n-1}), \\ I\pi &= \{p(x) \in \mathbf{Z}[x]/(x^n - 1) : p(1) = 0\}.\end{aligned}$$

A map  $I\pi \rightarrow \mathbf{Z}\pi/N$  is given by an assignment  $x^i - 1 \rightarrow p_i(x)$ ,  $1 \leq i \leq n-1$ , subject to the condition  $x^j p_i = p_{j+i} - p_j$ . Given  $p_1$ , we find  $p_2 = (1+x)p_1$ ,  $p_3 = (1+x+x^2)p_1, \dots$ , so the map is determined by  $p_1$ , and  $\text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N) \cong \mathbf{Z}\pi/N$ . Changing this map by the restriction of a map  $\mathbf{Z}\pi \rightarrow \mathbf{Z}\pi/N$  modifies  $p_1$  by  $(x-1)q$ ,  $q \in \mathbf{Z}\pi/N$ , so that

$$\begin{aligned}\text{Hom}_{\mathbf{Z}\pi}(I\mathbf{Z}_n, \mathbf{Z}\mathbf{Z}_n/N)/\mathbf{Z}\mathbf{Z}_n/N \\ \cong \mathbf{Z}[x]/(x^n - 1, 1 + x + \cdots + x^{n-1}, 1 - x) \cong \mathbf{Z}_n.\end{aligned}$$

The map determined by  $p_1 = x$ , namely  $p_i = x + \cdots + x^i$ , maps to a generator of  $H_1(\mathbf{Z}_n)$ . This map is precisely the map  $X$  described in the discussion of lens spaces.

It should be clear by now that  $\text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N)$  is crucially involved in the question of whether the two forms on  $I\pi \oplus \mathbf{Z}\pi/N$  are equivariantly isometric, but it seems difficult to write down the typical element of  $\text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N)$ , except for cyclic groups. We plan to circumvent this problem by tensoring with  $\mathbf{Z}[1/n]$  (or  $\mathbf{Q}$ ), since then  $I\pi$  and  $\mathbf{Z}\pi/N$  become isomorphic. But first, we rephrase our problem in more convenient terms.

So let  $\pi$  be any finite group of order  $n$ . In the usual bases, let  $(\ , \ )$  be the form on  $I\pi \oplus \mathbf{Z}\pi/N$  represented by  $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ , and let  $(\ , \ )'$  be represented by

$$\left( \begin{array}{cccc|c} 2 & 1 & \cdots & 1 & I \\ 1 & 2 & & & \\ \vdots & & \ddots & & \\ 1 & & & 2 & \\ \hline & I & & & 0 \end{array} \right).$$

LEMMA 4.2. *The isometries (equivariant or otherwise) of  $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$  are given by  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ , where*

$$(3) \quad A'C + C'A = 0, \quad B'D + D'B = 0, \quad A'D + C'B = I. \quad \square$$

We have that  $\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$  is an  $(\mathbf{Z})$  isometry from

$$(I\pi \oplus \mathbf{Z}\pi/N, (\ , \ )') \quad \text{to} \quad (I\pi \oplus \mathbf{Z}\pi/N, (\ , \ )).$$

Thus, every  $(\mathbf{Z})$  isometry

$$H: (I\pi \oplus \mathbf{Z}\pi/N, (\ , \ )) \rightarrow (I\pi \oplus \mathbf{Z}\pi/N, (\ , \ )')$$

satisfies  $\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ , with  $A, B, C, D$  as in (3). We conclude that every  $(\mathbf{Z})$  isometry  $H$  is given by

$$H = \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ -XA + C & -XB + D \end{pmatrix},$$

where  $A, B, C, D$  satisfy (3).

We would like to know whether  $H$  can be equivariant, i.e. do  $A, B, -XA + C, -XB + D$  represent  $\mathbf{Z}\pi$ -maps? This seems too difficult. In view of the geometric discussion above (and §§6, 7), define a *special isometry* to be an *equivariant isometry*  $H$ , as above, with  $A = \pm I$ .

More generally, for a closed 3-manifold  $M^3$ , we have  $\pi_2(s(M)) \cong \pi_2(s'(M)) \cong I\pi \oplus \pi_2(M_0)$  by Proposition 2.1. Define a *special isometry*

$$H: (I\pi \oplus \pi_2(M_0), (\ , \ )) \rightarrow (I\pi \oplus \pi_2(M_0), (\ , \ ))'$$

to be an equivariant isometry represented by  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ , with  $A = \pm I$ . The point of special isometries is given by

**PROPOSITION 4.3.** *A homotopy equivalence of pairs,  $f: (s(M), S^2 \times \{0\}) \rightarrow (s'(M), S^2 \times \{0\})$ , inducing the identity on  $\pi_1$ , induces a special isometry on  $\pi_2$ .*

**PROOF.** We can assume that  $f$  restricts to a homotopy equivalence  $f: M_0 \times S^1 \rightarrow M_0 \times S^1$ , inducing  $\epsilon\tau$  on  $\partial(M_0 \times S^1)$ , where  $\epsilon$  corresponds to possible orientation reversals of  $S^2$  and  $S^1$ . Lifting to  $\tilde{f}: \tilde{M}_0 \times S^1 \rightarrow \tilde{M}_0 \times S^1$ , we see that  $\tilde{f}$  preserves the boundaries of the annuli  $\gamma_g \times S^1$  of Proposition 2.1 and either preserves all the orientations, or reverses them all. Referring to the proof of Proposition 2.1, we see that  $\tilde{f}$  induces a map between the exact sequences giving  $\pi_2(s(M))$  and  $\pi_2(s'(M))$ , and the result follows. In fact, the induced special isometry is given by  $\begin{pmatrix} \pm I & 0 \\ \pm I & \pm I \end{pmatrix}$ .  $\square$

We will eventually (Theorem 7.4) reverse this implication, and show that the existence of a special isometry allows us to extend  $\tau: \partial M_0 \times S^1 \rightarrow \partial M_0 \times S^1$  to a homotopy equivalence of  $M_0 \times S^1$ , thereby giving a homotopy equivalence of pairs.

Returning to the algebraic question arising from the two forms defined on  $I\pi \oplus \mathbf{Z}\pi/N$ ,  $\pi$  any finite group, we ask whether integral special isometries exist. We emphasize integral, since we can tensor with a commutative ring, say  $\mathbf{Z}[1/n]$  or  $\mathbf{Q}$ , and ask the same question. The main result of this section is

**THEOREM 4.4.** *Special isometries  $(I\pi \oplus \mathbf{Z}\pi/N, (\ , \ )) \rightarrow (I\pi \oplus \mathbf{Z}\pi/N, (\ , \ ))'$  exist over  $\mathbf{Z}[1/n]$  for any finite group  $\pi$  of order  $n$ . An integral special isometry exists if and only if there is a homomorphism  $\psi: \pi \rightarrow \mathbf{Z}[1/n]/\mathbf{Z}$  such that  $\psi(g) = \frac{1}{2}$  for all elements  $g$  of order 2.*

**PROOF.** It is well known that  $I\pi$  and  $\mathbf{Z}\pi/N$  become isomorphic if we invert  $|\pi| = n$ . To see this, let  $R = \mathbf{Z}[1/n]$ , and let  $I_R\pi = \ker\{R\pi \xrightarrow{\epsilon} \mathbf{Z}\}$ . The natural map

$$I_R\pi \xrightarrow{\mathbf{Z}} R\pi/N,$$

$$g - e \rightarrow g - e \equiv g + \sum_{h \neq e} h$$

has inverse

$$R\pi/N \xrightarrow{\mathbf{Z}^{-1}} I_R\pi,$$

$$g \rightarrow g - \frac{N}{n} = (g - e) - \frac{1}{n} \sum_{h \neq e} (h - e).$$

In terms of the usual bases, we have

$$Z = I + \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad Z^{-1} = I - \frac{1}{n} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$

This allows us to rewrite  $H$  as

$$H = \left( \begin{array}{c|c} A & BZ \\ \hline Z^{-1}(-XA + C) & Z^{-1}(-XB + D)Z \end{array} \right),$$

representing a map  $I_R\pi \oplus I_R\pi \rightarrow I_R\pi \oplus I_R\pi$ , with the basis  $\{g - e\}_{g \neq e}$  used for both factors. The original  $H$  is equivariant if and only if the new  $H$  is equivariant.

Now assume  $A = \pm I$ . Replacing  $H$  by  $-H$  if necessary, we can assume  $A = I$ . From (3),  $C = -C'$ .

Assume  $H$  is equivariant, and consider the lower left entry of  $H$ ,  $Z^{-1}(-X + C)$ . By Lemma 3.3, extended in the obvious way, this map has the form

$$(4) \quad h - e \rightarrow \sum_{g \neq e} m_g [(hg - e) - (h - e) - (g - e)] + m(h - e) \\ = - \left[ \sum_{g \neq e, h} (m_g - m_{h^{-1}g})(g - e) + \left\{ \sum_{g \neq e} m_g + m_h - m \right\} (h - e) \right]$$

for  $m_g, m \in \mathbb{Z}[1/n]$ . This expression determines the  $h$ th column of  $Z^{-1}(-X + C)$ .

Observe that multiplying a matrix on the left by  $Z$  adds the sum of a given column to every entry in that column. The  $h$ th column of  $Z^{-1}(-X + C)$  sums to

$$- \left[ \sum_{g \neq e, h} (m_g - m_{h^{-1}g}) + \sum_{g \neq e} m_g + m_h - m \right] = - \left[ \sum_{g \neq e} m_g + m_{h^{-1}} - m \right].$$

Let  $\bar{m} = \sum_{g \neq e} m_g$ . Then the  $h$ th column of  $-X + C$  is given by

$$(5) \quad h - e \rightarrow - \left[ \sum_{g \neq e, h} (m_g - m_{h^{-1}g} + m_{h^{-1}} + \bar{m} - m)g \right. \\ \left. + \{2(\bar{m} - m) + m_h + m_{h^{-1}}\}h \right].$$

Since  $C + C' = 0$ , we have

$$(-X + C) + (-X + C)' = -(X + X') = - \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & & \\ \vdots & & \ddots & \\ 1 & & & 2 \end{pmatrix}.$$

Using (5) to compute  $(-X + C) + (-X + C)'$ , we find

$$(6a) \quad m_g - m_{h^{-1}g} + m_{h^{-1}} + m_h - m_{g^{-1}h} + m_{g^{-1}} + 2(\bar{m} - m) = 1, \quad g \neq h,$$

$$(6b) \quad m_g + m_{g^{-1}} + 2(\bar{m} - m) = 1, \quad g \neq e.$$

In particular,  $m_g + m_{g^{-1}}$  is a constant. Notice that (6a) follows from (6b). The equivariance of  $H$  is thus equivalent to choosing  $m_g, m \in \mathbb{Z}[1/n]$  so that (6b) holds.

It is a simple exercise to show that this is equivalent to

$$(7a) \quad m_g + m_{g^{-1}} = \frac{1 + 2m}{n},$$

$$(7b) \quad \bar{m} = \frac{(n-1)(1+2m)}{2n},$$

so we can always choose  $m_g \in \mathbf{Z}[1/n]$  to satisfy (6b). To construct a special isometry, simply let  $H = \begin{pmatrix} I & 0 \\ C-X & I \end{pmatrix}$ , where  $C-X$  is given by (5) and (7). In general, the entries lie in  $\mathbf{Z}[1/n]$ .

Using (5) and (7),  $C-X$  is given by

$$(8) \quad h - e \rightarrow - \left[ \sum_{g \neq e, h} \left( m_g + m_{h^{-1}g} - m_{h^{-1}g} + \frac{n-1-2m}{2n} \right) g + h \right].$$

Thus, we have reduced the problem of finding an integral special isometry to the problem of choosing  $m_g$ ,  $m \in \mathbf{Z}[1/n]$ , so that (7) holds and the entries in (8) are integers. The following lemma now completes the proof of Theorem 4.4, and also provides a method for constructing integral special isometries.

LEMMA 4.5. *Let  $\pi$  be a finite group of order  $n$ . Then there exist  $m_g$ ,  $m \in \mathbf{Z}[1/n]$ ,  $g \neq e$ , so that*

$$(9) \quad \begin{aligned} m_g + m_{g^{-1}} &= \frac{1+2m}{n}, \\ m_g + m_{h^{-1}g} - m_{h^{-1}g} + \frac{n-1-2m}{2n} &\in \mathbf{Z}, \quad g \neq h, \end{aligned}$$

$$(10) \quad \begin{aligned} &\text{if and only if there exists } \psi: \pi \rightarrow \mathbf{Z}[1/n]/\mathbf{Z} \text{ so that } \psi(g) = \frac{1}{2} \\ &\text{for all } g \text{ of order } 2. \end{aligned}$$

PROOF. Suppose that  $m_g$ ,  $m$  exist. Define  $\psi: \pi \rightarrow \mathbf{Z}[1/n]/\mathbf{Z}$  by  $\psi(e) = 0$ ,  $\psi(g) = m_g + (n-1-2m)/2n$ . Then (7a) and (9) show that  $\psi$  is a homomorphism, and if  $g = g^{-1}$ ,  $\psi(g) = \frac{1}{2}$ .

Conversely, given  $\psi$ , define  $m_g$ ,  $m$  as follows: Let  $m \in \mathbf{Z}[1/n]$  be arbitrary. If  $g^2 = e$ ,  $m_g = (1+2m)/2n$ . If  $g^2 \neq e$ , let  $m_g = \tilde{\psi}(g) - (n-1-2m)/2n$ , where  $\tilde{\psi}(g)$  is a lift of  $\psi(g)$  to  $\mathbf{Z}[1/n]$ , and where  $m_g + m_{g^{-1}} = (1+2m)/n$ . This is possible since  $\tilde{\psi}(g) + \tilde{\psi}(g^{-1}) \in \mathbf{Z}$ .

Finally, note that we can always arrange  $m = 0$ . For, given choices  $m_g$ ,  $m$  as above, let  $m'_g = m_g - m/n$ ,  $m' = 0$ . Then  $m'_g$ ,  $m'$  satisfy (7a) and (9), and (8) is unchanged. Thus, every special isometry can be constructed as above, with  $m = 0$ .

□

REMARK. (1) If  $\pi$  has odd order, integral special isometries exist.

(2) Let  $\pi = \mathbf{Z}_n$ , generated by  $g$ . Take  $\psi: \mathbf{Z}_n \rightarrow \mathbf{Z}[1/n]/\mathbf{Z}$  to be  $\psi(g^i) = -i/n$ . Lift  $\psi$  as in Lemma 4.5 to obtain  $m_{g^i} = \frac{1}{2} + 1/2n - i/n$ . Retracing through the proof of Theorem 4.4, we recover the isometry  $\begin{pmatrix} I & 0 \\ -X & I \end{pmatrix}$  arising from the geometry of lens spaces.

(3) More generally, we should consider special isometries covering an automorphism  $\alpha \in \text{Aut } \pi$ . In this case,  $A = I$  is replaced by the permutation matrix  $(\delta_{\alpha(g),h})$ . However, composing such a special isometry with the  $\alpha^{-1}$ -isomorphism

$$\left( \begin{array}{c|c} \delta_{\alpha^{-1}(g),h} & 0 \\ \hline 0 & \delta_{\alpha^{-1}(g),h} \end{array} \right)$$

reduces the problem to isometries covering the identity. Thus, Theorem 4.4 holds in this more general setting.

We now investigate the curious condition (10) of Lemma 4.5.

**LEMMA 4.6.** *Suppose there exists  $\psi: \pi \rightarrow \mathbf{Q}/\mathbf{Z}$  with  $\psi(g) = \frac{1}{2}$  for all  $g^2 = e$ . Then each element of order 2 is nontrivial in  $H_1(\pi)$ .*

**PROOF.** Obvious, since  $\psi$  factors through abelianization.  $\square$

For example, the dihedral group  $D_{2m} = (x, y \mid y^m = x^2 = 1, xyx = y^{-1})$ ,  $m$  odd, satisfies (10), since  $H_1(D_{2m}) \cong \mathbf{Z}_2(x)$ , and all elements of order 2 are conjugate to  $x$ . The implication of Lemma 4.6 cannot be reversed—consider  $\mathbf{Z}_2 \times \mathbf{Z}_2$ . On the other hand, if  $\pi$  has a *unique* element  $g^*$  of order 2, then clearly (10) is equivalent to  $g^* \neq 0 \in H_1(\pi)$ .

Now assume that  $\pi$  acts freely on a homotopy sphere  $\Sigma^m$ . A theorem of Milnor [20] shows that  $\pi$  has at most one element of order 2.

**PROPOSITION 4.7.** *Let  $\pi$  act freely on  $\Sigma^m$ , and let  $g^*$  be the unique element of order two (if  $n$  is even). Then the condition that there exists a map  $\psi: \pi \rightarrow \mathbf{Q}/\mathbf{Z}$  with  $\psi(g^*) = \frac{1}{2}$  is equivalent to the condition that all Sylow subgroups of  $\pi$  are cyclic.*

**PROOF.** Since  $\pi$  acts freely on the homotopy sphere  $\Sigma^m$ ,  $\pi$  has periodic cohomology, so that the odd Sylow subgroups are cyclic, and the 2-Sylow is either cyclic or generalized quaternion [20]. We write generalized quaternion groups as  $Q_{4 \cdot 2^k} = (x, y \mid x^2 = (xy)^2 = y^{2^k})$ , with  $k = 1$  corresponding to the usual eight element quaternion group.

Notice that  $Q_{4 \cdot 2^k}$  abelianizes to  $\mathbf{Z}_2(x) \times \mathbf{Z}_2(y)$ , but the element of order two,  $x^2$ , maps trivially. Therefore, if the 2-Sylow of  $\pi$  is quaternion,  $\psi$  cannot exist.

Conversely, if the 2-Sylow is cyclic, a theorem of Burnside [34, p. 163] shows that  $\pi$  is metacyclic (type I),  $\pi = (A, B \mid A^m = B^{\bar{n}} = 1, BAB^{-1} = A^r)$ , where  $r^{\bar{n}} \equiv 1 \pmod{m}$  and  $((r-1)\bar{n}, m) = 1$ . We assume  $|\pi| = n = m\bar{n}$  is even, since otherwise the result is obvious. Now  $m$  cannot be even, since then  $r$  is even, contradicting  $r^{\bar{n}} \equiv 1 \pmod{m}$ . So  $\bar{n}$  is even, and now  $H_1(\pi) \cong \mathbf{Z}_{\bar{n}}$ , generated by  $B$ , and the element of order two goes nontrivially.  $\square$

Finally, consider those  $\pi$  that act freely on a homotopy 3-sphere. All known examples arise from orthogonal actions on  $S^3$ , and these give either lens spaces or Seifert manifolds  $\Sigma(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ , where  $b \in \mathbf{Z}$ ,  $(\alpha_i, \beta_i) = 1$ ,  $0 < \beta_i < \alpha_i$ ,  $i = 1, 2, 3$ , and  $\{\alpha_1, \alpha_2, \alpha_3\}$  is a Platonic triple  $\{2, 2, \alpha\}$ ,  $\{2, 3, 3\}$ ,  $\{2, 3, 4\}$ , or  $\{2, 3, 5\}$  [25]. The manifolds corresponding to  $\{2, 2, \alpha\}$  are the so-called prism manifolds.

Except for cyclic groups, the only groups with even abelianization are those corresponding to  $\{2, 3, 4\}$  and  $\{2, 2, \alpha\}$ . The groups corresponding to  $\{2, 3, 4\}$  and  $\{2, 2, \alpha\}$ ,  $\alpha$  even, have generalized quaternion 2-Sylow subgroup. If  $\alpha$  is odd, however,  $\pi$  has a cyclic 2-Sylow. Following [25], we write these as

$$\begin{aligned} D_{4\alpha}^* &= (x, y \mid x^2 = (xy)^2 = y^\alpha), & \alpha \text{ odd}, \\ D_{2^k\alpha}^* &= (x, y \mid x^{2^k} = 1 = y^\alpha, xyx^{-1} = y^{-1}), & \alpha \text{ odd}, k \geq 2, \end{aligned}$$



together with the product of one of these with a cyclic group of relatively prime order. Note that  $D'_{2^k\alpha}$  is obviously metacyclic, and  $D_{4\alpha}^*$  can be rewritten as  $(A, B \mid A^\alpha = B^4 = 1, BAB^{-1} = A^{-1})$ , where  $A = yx^2$ ,  $B = x$ . The correspondence between these groups and Seifert manifolds can be found in [25, p. 112].

Finally, there is a list of groups which might conceivably act on a homotopy 3-sphere [25, p. 113; 16], but they all have generalized quaternion 2-Sylow subgroups.

Combining this discussion with Theorem 4.4 and Proposition 4.7, we have

**COROLLARY 4.8.** *Suppose  $\pi$  acts freely on a homotopy 3-sphere  $\Sigma$ . Then there is a special isometry  $\pi_2(s(\Sigma/\pi)) \rightarrow \pi_2(s'(\Sigma/\pi))$  if and only if all Sylow subgroups of  $\pi$  are cyclic. The groups which satisfy this condition are  $\mathbf{Z}_n$ ,  $D_{4\alpha}^*$  ( $\alpha$  odd),  $D'_{2^k\alpha}$  ( $\alpha$  odd,  $k \geq 2$ ), and the direct product of one of these with a cyclic group of relatively prime order.  $\square$*

**5. Spinning 3-manifolds with no 2-sided projective planes.** We now treat the case of an arbitrary closed 3-manifold  $M$  with no 2-sided projective planes. We do not believe this restriction is necessary, but we have not pursued the question here, since our main interest is connected sums of aspherical and spherical manifolds (see §6). By well-known 3-manifold theory [9],  $M$  has a connected sum decomposition, with summands either aspherical, spherical,  $S^2 \times S^1$ , or  $S^2 \tilde{\times} S^1$ . To describe  $\pi_2(M_0)$ , we follow the discussion and notation in [10].

Let  $M = (\#_{i=1}^m M_i) \# (\#_{j=1}^l M_{m+j})$  be a prime decomposition for  $M$ , where  $M_i$ ,  $i = 1, \dots, m$ , has  $\pi_2 = 0$ , and  $M_{m+j}$ ,  $j = 1, \dots, l$ , is  $S^2 \times S^1$  or  $S^2 \tilde{\times} S^1$ . An Epstein system for  $M_0$  is a disjoint collection of embedded 2-spheres which generate  $\pi_2(M_0)$  as a  $\mathbf{Z}\pi$ -module. There are many such Epstein systems; we choose one as in Figure 2. The 2-spheres  $S_1, \dots, S_{m+l}$  cut  $M_0$  into prime summands, and the  $S_{m+l+1}, \dots, S_{m+2l}$  are 2-sphere fibers of the  $M_{m+1}, \dots, M_{m+l}$ .

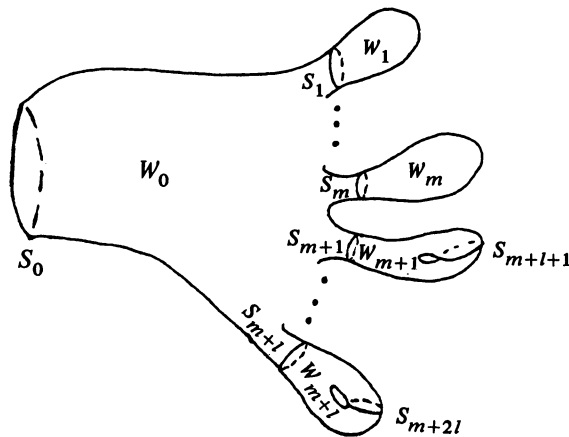


FIGURE 2

Let  $S = \bigcup_{i=0}^{m+2l} S_i$ . Let  $p: \tilde{M}_0 \rightarrow M_0$  be the universal cover, and set  $S^* = p^{-1}(S)$ . The long exact sequence for  $(\tilde{M}_0, S^*)$  yields an exact sequence

$$0 \rightarrow H_3(\tilde{M}_0, S^*) \rightarrow H_2(S^*) \rightarrow \pi_2(M_0) \rightarrow 0.$$

Letting  $\mathcal{M} = \{W: W \text{ is a connected component of } M_0 - S, \text{ with } \overline{W} - W \subset S, \text{ and } |\pi_1 W| < \infty\}$ , we have [10, Lemma, p. 110]

$$H_3(\tilde{M}_0, S^*) = \bigoplus_{W \in \mathcal{M}} H_3(p^{-1}\overline{W}, p^{-1}(\overline{W} - W)).$$

For instance,  $W_0$  is a punctured sphere, so its lifts give relations among the lifts of  $S_0, \dots, S_{m+l}$ . Similarly, if  $M_i$  is spherical, the lifts of  $W_i$  are punctured homotopy spheres, so give relations among the lifts of  $S_i$ , as in §4. And similarly for the  $M_{m+j}$ .

We now describe intersection forms for  $s(M)$  and  $s'(M)$ . On  $I\pi$ , Proposition 2.2 applies, and the forms are obviously zero on  $\pi_2 M_0$ . On “mixed pieces,” the forms are identical, given as follows: Pick a lift  $\tilde{*}$  of  $*$ , and let  $\tilde{W}_0$  be the lift of  $W_0$  containing  $\tilde{*}$ , with  $\partial\tilde{W}_0 = \tilde{S}_0 \cup \dots \cup \tilde{S}_{m+l}$ . Also pick the obvious lifts of  $S_{m+l+1}, \dots, S_{m+2l}$ . Orient the  $\tilde{S}_0, \dots, \tilde{S}_{m+l}$  as the boundary of  $\tilde{W}_0$ , orient the  $S_{m+l+1}, \dots, S_{m+2l}$ , and transport these via covering transformations to  $(S_i)_g$ ,  $g \in \pi$ . Now let  $g = g_1 \cdots g_n$  be written in reduced form, with  $g_k \in \pi_1 M_{i_k}$ , represented by a path  $\sigma$ . Then

$$S_{g-e} \cdot (S_0)_e = -1, \quad S_{g-e} \cdot (S_0)_h = \omega(g) \delta_{g,h}, \quad h \neq e.$$

Also, as  $\sigma$  enters and leaves lifts of the  $W_i$ ,  $\sigma$  pierces the lifts of  $S_i$ , so intersections between  $S_{g-e}$  and lifts of  $S_{i_k}$  are introduced. Finally, if  $m+1 \leq i_k \leq m+l$ , we also find intersections with lifts of  $S_i$ ,  $m+l+1 \leq i \leq m+2l$ . We spare the reader the notation necessary to write this explicitly, since we will not need it.

We can now generalize Theorem 4.4.

**THEOREM 5.1.** *Let  $M^3$  be a closed 3-manifold with no 2-sided  $P^2$ 's. There is a special isometry  $H: (\pi_2(s(M)), (\cdot, \cdot)) \rightarrow (\pi_2(s'(M)), (\cdot, \cdot))$  if and only if every summand of  $M$  is  $S^2 \times S^1$ ,  $S^2 \tilde{\times} S^1$ , or  $\Sigma^3/\pi$ , where all Sylow subgroups of  $\pi$  are cyclic.*

**PROOF.** Suppose some summand, say  $M_1$ , is either aspherical or  $\Sigma/\pi$ , where  $\pi$  has a generalized quaternion 2-Sylow subgroup. Let  $g \in \pi_1(M_1)$ . Then  $S_{g-e}$  has very simple intersections—namely

$$\begin{aligned} S_{g-e} \cdot (S_0)_e &= -1, & S_{g-e} \cdot (S_0)_g &= \omega(g), \\ S_{g-e} \cdot (S_1)_e &= 1, & S_{g-e} \cdot (S_1)_g &= -\omega(g), \end{aligned}$$

and all other intersections with generators of  $\pi_2(M_0)$  are zero. This means that, in computing intersections, we can ignore lifts of all spheres, except  $S_0$  and  $S_1$ .

More precisely, let  $N$  be the  $\mathbf{Z}\pi$ -submodule of  $\pi_2(M_0)$  generated by  $S_2 \cup \dots \cup S_{m+2l}$ . Then  $\pi_2(M_0)/N$  is naturally isomorphic to a direct sum of copies of  $\pi_2((M_1)_0)$ , indexed by the cosets  $\pi/\pi_1(M_1)$ , i.e.  $\pi_2(M_0)/N \cong \mathbf{Z}\pi \otimes_{\mathbf{Z}\pi_1(M_1)} \pi_2((M_1)_0)$ . In fact, since  $S_{g-e}$  only intersects elements from  $e \otimes \pi_2(M_1)_0$  (note that  $S_0 = -S_1$  in  $\pi_2(M_0)/N$ ), we may further project to  $\pi_2((M_1)_0)$ . This induces  $\bar{H}: I\pi_1(M_1) \oplus \pi_2((M_1)_0) \rightarrow I\pi_1(M_1) \oplus \pi_2((M_1)_0)$ . Now  $\bar{H}$  may not be a special isometry, but  $\bar{H}$

does have the form  $(\begin{smallmatrix} \pm I & B \\ C & D \end{smallmatrix})$ , and  $\bar{H}|_{I_{m_1}(M_1)}$  is an isometry. This contradicts the proof of either Theorem 3.1 or Theorem 4.4/Proposition 4.7.

Conversely, suppose every summand of  $M$  is as in Theorem 5.1. By Case 3, §2, and Corollary 4.8, each summand admits special isometries. It is straightforward to put these together to obtain a special isometry for  $M$  (see §7).  $\square$

REMARK. As stated, Theorem 5.1 applies to special isometries covering the identity map on  $\pi_1$ . With a bit more work, Theorem 5.1 also works for maps covering automorphisms of  $\pi_1$  (see Remark (3) following Lemma 4.5).

**6. Knots in  $S^4$  with the same complements.** We are interested here in fibered knots in  $S^4$  with periodic monodromy. We first recall the twist spun knots of Zeeman [35]. Let  $K$  be a smooth knot in  $S^3$ , and let  $M_k$  be the  $k$ -fold cyclic branched cover of  $S^3$ , branched along  $K$ , with canonical branched covering transformation  $\sigma$ . Then the  $k$ -twist spin of  $K$ ,  $K_k$ , is a fibered knot in  $S^4$ , with fiber  $(M_k)_0$  and monodromy  $\sigma$ , so  $X(K_k) = (M_k)_0 \times_{\sigma} S^1$ .

More generally, if  $0 < p < k$ ,  $(k, p) = 1$ , then the  $p$ -fold cyclic branched cover of  $K_k$ , say  $K_{k,p}$ , is again a knot in  $S^4$ , with exterior  $X(K_{k,p}) = (M_k)_0 \times_{\sigma^p} S^1$  [27]. The associated knots  $K_{k,p}^*$  are again knots in  $S^4$ . This was proved by Gordon for  $p = 1$  [8], and by Pao in general [27]. Pao's description of  $K_{k,p}$  used the natural  $S^1$  action associated to a bundle over  $S^1$  with periodic monodromy, together with Fintushel's work [5] on  $S^1$  actions on 4-manifolds. Together, their work gives

**PROPOSITION 6.1.** *Modulo the 3-dimensional Poincaré conjecture, the class of all fibered knots in  $S^4$  with periodic monodromy is precisely the class of all  $k$ -twist spin knots and their  $p$ -fold cyclic branched covers,  $0 < p < k$ ,  $(k, p) = 1$ .*

**PROOF (SKETCH).** Let  $(S^4, S^2)$  be fibered with periodic monodromy, so that  $S^4 - S^2 \times \dot{D}^2 = Y \times_{\beta} S^1$ ,  $\beta^k = 1$ . The natural  $S^1$  action on  $Y \times_{\beta} S^1$  has a punctured homotopy 3-sphere as orbit space, with a knotted arc as the image of an annulus of exceptional orbits with stabilizer  $\mathbf{Z}_k$  and slice representation given by rotation of a normal disk by  $2\pi p/k$ . Hence,  $(Y, \text{Fix}(\beta))$  is the punctured  $k$ -fold cyclic branched cover of a knot in a homotopy 3-sphere, and  $\beta = \sigma^p$ . If the homotopy sphere is  $S^3$ , the knot is  $K_{k,p}$ , where  $K$  is the knot associated to the knotted arc in the orbit space. See [5, 27] for details.  $\square$

We now give some simple limitations on the possible 3-manifolds which are cyclic branched covers of knots in homotopy 3-spheres. Let  $M$  be such a manifold. Of course,  $M$  is orientable. An easy argument using the equivariant sphere theorem of Meeks and Yau [19] shows that (i)  $M$  has no  $S^2 \times S^1$  summands, and (ii) if  $M$  is a connected sum,  $M$  splits equivariantly as the cyclic branched cover of a connected sum of knots, so we may reduce to irreducible summands [30].

Suppose  $M = \Sigma/\pi$ . We claim  $\pi$  cannot be metacyclic, unless  $\pi$  is actually cyclic of odd order. To see this, observe that if  $\pi$  is not cyclic, then  $H_1(\pi) \cong \mathbf{Z}_{\bar{n}}$ ,  $\bar{n}$  even (see Proposition 4.7). The action of the monodromy on  $H_1(\pi)$  is multiplication by  $s$ ,  $(s, \bar{n}) = 1$ , and the Wang sequence of the fibration shows that  $\cdot(s-1)$  is an isomorphism. But this forces  $s$  to be both even and odd. Finally, if  $\pi$  is cyclic of odd

order,  $\cdot(-1)$  is the only possible monodromy, and of course this is realized by lens spaces as 2-fold cyclic branched covers of 2-bridge knots. For a complete description of monodromies of fibered knots in  $S^4$  with punctured spherical space form fibers, see [32].

**THEOREM 6.2.** *Let  $K$  be any nontrivial fibered knot in  $S^4$  with odd order monodromy. Then  $K$  is not determined by its complement.*

**PROOF.** Let  $K$  have fiber  $M_0$  and monodromy  $\sigma$  of odd order  $k$  so that

$$K = (M_0 \times_{\sigma} S^1 \cup_{\text{id}} S^2 \times D^2, S^2 \times \{0\}),$$

$$K^* = (M_0 \times_{\sigma} S^1 \cup_{\tau} S^2 \times D^2, S^2 \times \{0\}).$$

If  $K$  and  $K^*$  are equivalent, there is a homeomorphism of  $M_0 \times_{\sigma} S^1$  extending  $\varepsilon\tau$  on  $\partial M_0 \times_{\sigma} S^1$ . Lifting to the  $k$ -fold cyclic cover, and using that  $k$  is odd, we find a homeomorphism of  $M_0 \times S^1$  extending  $\varepsilon\tau$  on  $\partial M_0 \times S^1$ . As in Proposition 4.3, this induces a special isometry on  $\pi_2(s(M))$ , possibly covering an automorphism of  $\pi$ . But every summand of  $M$  is either aspherical or, by the discussion above, spherical with generalized quaternion 2-Sylow. This contradicts Theorem 5.1 (or the remark following Theorem 5.1).  $\square$

**REMARKS.** (1) This generalizes [8, Proposition 4.2], where Gordon proves that a fibered knot in  $S^4$  ( $S^n$ ) with (i) odd order monodromy, and (ii) the unpunctured fiber has  $\mathbf{R}^3$  ( $\mathbf{R}^{n-1}$ ) as universal cover, is not determined by its complement. It also generalizes [32], where the author and A. Suciu handle the spherical fibers not covered by Gordon's theorem—the binary icosohedral space  $S^3/\text{SL}(2, 5)$  and the quaternion manifold  $S^3/Q_8$ . Both of these obstruction theoretic proofs strongly use that the universal covers are  $\mathbf{R}^3$  or  $S^3$  (no counterexamples are known). Theorem 6.2 shows that this is irrelevant, and also handles all connected sums.

(2) If the fiber of  $K$  is a punctured aspherical 3-manifold, the proof of Theorem 6.2, and Theorem 3.1, show that not only are  $K$  and  $K^*$  distinct knots with the same complement, but they have  $k$ -fold cyclic branched covers which are not even homotopy equivalent!

(3) Suppose  $K$  is a fibered knot with even order monodromy. If the order is two,  $K$  is determined by its complement (Litherland [8, p. 595; 22, 31]), but nothing is known otherwise. If the order has an odd factor  $k$ , one might try to mimic the proof of Theorem 6.2 by lifting to the  $k$ -fold cover. These covers are not as simple as in Theorem 6.2, where the fibration over  $S^1$  becomes a product. Nevertheless, it seems reasonable to conjecture that  $K$  is not determined by its complement. In fact, I would conjecture that this is the case for all fibered knots in  $S^4$  with monodromy of finite order greater than two.

**7. Rotations in 2-spheres.** Let  $M$  be a closed 3-manifold, and recall the rotation in  $\partial M_0$ ,  $\rho$ , as defined in §1. The connection between  $\rho$  and the twist  $\tau$  is given by the following simple observation.

**LEMMA 7.1.** *If  $\rho \simeq \text{id}_{M_0} \text{ (rel } \partial)$ , then  $\tau: \partial M_0 \times S^1 \rightarrow \partial M_0 \times S^1$  extends to a (fiber preserving) homotopy equivalence of  $M_0 \times S^1$ .*

PROOF. Let  $F: M_0 \times [-1, 0] \rightarrow M_0$  be a homotopy (rel  $\partial$ ) with  $F_{-1} = \text{id}_{M_0}$ ,  $F_0 = \rho$ . Let  $G: M_0 \times [0, 1] \rightarrow M_0$  be a homotopy, supported in a collar of  $\partial M_0$ , with  $G_0 = \rho$ ,  $G_1 = \text{id}_{M_0}$ , and  $G(x, t) = \rho_{2\pi t}(x)$ ,  $x \in \partial M_0$ . Then  $F$  and  $G$  fit together to give the desired extension of  $\tau$ .  $\square$

COROLLARY 7.2. *If  $\rho \simeq \text{id}_{M_0}$  (rel  $\partial$ ), then every summand of  $M$  is either  $S^2 \times S^1$ ,  $S^2 \tilde{\times} S^1$ ,  $\simeq P^2 \times S^1$ , or  $\Sigma/\pi$ , where all Sylow subgroups of  $\pi$  are cyclic.*

PROOF. Since  $\tau$  extends to  $M_0 \times S^1$ , Proposition 4.3 provides a special isometry of  $\pi_2$ , and the result follows from Theorem 5.1 if  $M$  has no 2-sided  $P^2$ 's. Otherwise, we appeal to Hendriks [10].

We now begin to reverse the implication of Corollary 7.2. The following lemma will allow us to reduce to the prime summands of  $M$ .

LEMMA 7.3. *Let  $W_0$  be a punctured 3-ball, with  $\partial W_0 = \bigcup_{i=0}^m S_i$ . Then the rotation  $\rho_0$  in  $S_0$  is homotopic (rel  $\partial$ ) to the disjoint union of the rotations in the other boundary spheres  $\sqcup_{i=1}^m \rho_i$  [10, Remarque, (iii), p. 182].*

PROOF. Use the rotational symmetry of  $W_0$  to untwist  $S_0$ , thereby twisting  $\bigcup_{i=1}^m S_i$ .  $\square$

It should be fairly clear that rotations in 2-spheres play an important role in the description of homotopy equivalences of 3-manifolds. For instance, the (based) homotopy equivalences of aspherical summands are given by  $\text{Aut}(\pi)$ , and the (based) homotopy equivalences of the spherical summands are given by  $\{\alpha \in \text{Aut}(\pi) \mid \alpha_*: H_3(\pi) \rightarrow H_3(\pi) \text{ is } \pm 1\}$  [23, 29]. In this sense the pieces of a homotopy equivalence are understood, and the rotations tell us how the pieces are glued together.

Now, a complete theory has been worked out by Hendriks [10] (see also the work of McCullough [18]). We present here an alternative proof of a major component of that theory: When is  $\rho \simeq \text{id}_{M_0}$  (rel  $\partial$ )? Our contribution is to explicitly relate this problem to the special isometries of §4. As a result, when the homotopy exists, we can “understand” it, at least to the extent that we can explicitly describe the homotopy on the 1-cells of  $M_0$ .

Actually, it seems almost fitting that this question should be answered by 4-dimensional methods. Laudénbach [15, Appendix II] originally showed that a rotation in a *nonseparating* sphere is not homotopic to the identity by framing considerations. (For example, consider  $S^2 \times S^1$ , and let  $\rho$  be a rotation in a fiber. If  $\rho \simeq \text{id}$ , then  $S^2 \times T^2 \simeq (S^2 \times S^1) \times_\rho S^1$ . But the intersection forms are  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ .)

The first part of our proof follows Hendriks—we analyze the primary and secondary obstructions which arise as we try to homotop  $\rho$  to  $\text{id}_{M_0}$  (rel  $\partial$ ). Thus, we now briefly outline how the obstructions arise, and which cohomology/homotopy groups are relevant. We follow Hendriks' discussion and notation.

Let  $(X, A, x)$  be a relative CW-complex. Given two maps  $f, g: (X, x) \rightarrow (Y, y)$  with  $f|_{(X, A)^k} = g|_{(X, A)^k}$ ,  $k \geq 2$ , inducing  $\theta: \pi_1(X, x) \rightarrow \pi_1(Y, y)$ , one defines a cohomology class  $d^{k+1}(f, g; A) \in H^{k+1}(X, A; \theta^* \pi_{k+1}(Y))$ , the *primary obstruction*

to deforming  $f$  to  $g$  (rel  $A$ ). This is zero precisely when there exists  $g': X \rightarrow Y$  such that  $g' \simeq g$  (rel  $(X, A)^{k-1}$ ) and  $f|_{(X, A)^{k+1}} = g'|_{(X, A)^{k+1}}$ . Thus, we alter the constant homotopy from  $f$  to  $g$  on  $(X, A)^k \times I$  (by  $\eta \in C^k(X, A; \theta * \pi_{k+1}(Y))$ , where  $\delta\eta = d^{k+1}$ ), and the homotopy now extends to  $(X, A)^{k+1} \times I$ .

If  $d^{k+1}(f, g; A) \neq 0$ , we can consider a secondary obstruction by altering the homotopy on  $(X, A)^{k-1}$ . Define a homomorphism

$$\Delta^{k-1}(f; A): H^{k-1}(X, A; \theta * \pi_k(Y)) \rightarrow H^{k+1}(X, A; \theta * \pi_{k+1}(Y))$$

as follows [10, p. 104]: Let  $[\zeta] \in H^{k-1}(X, A; \theta * \pi_k(Y))$ . Construct a homotopy of  $f$  (rel  $(X, A)^{k-2}$ ),

$$G: X \times [0, 1] \rightarrow Y,$$

so that for each  $(k-1)$ -cell  $\sigma$  of  $(X, A)$ ,

$$d^k(G|_{\sigma \times [0, 1]}, f \circ \text{pr}_1|_{\sigma \times [0, 1]}; \partial(\sigma \times [0, 1])) = \zeta(\sigma),$$

and so that  $G_1|_{(X, A)^k} = f|_{(X, A)^k}$ . Then  $\Delta^{k-1}(f; A)([\zeta]) \equiv d^{k+1}(f, G_1; A)$ . In other words, homotop  $f$  (to itself) on  $(X, A)^{k-1}$  so as to “build in”  $[\zeta]$ , extend the homotopy of  $f$  (again to itself) to  $(X, A)^k$  (this is possible since  $\zeta$  is a cocycle), extend the homotopy to  $(X, A)^{k+1}$  (using a deformation retraction of  $e^{k+1} \times I$  to  $e^{k+1} \times \{0\} \cup \partial e^{k+1} \times I$ ), and take the primary obstruction from  $f$  to the result of the homotopy ( $G_1$ ) on  $(X, A)^{k+1}$ .

The significance of  $\Delta^{k-1}(f; A)$  is the following: Given  $f, g: (X, x) \rightarrow (Y, y)$  with  $f|_{(X, A)^k} = g|_{(X, A)^k}$ ,  $k \geq 2$ , there exists  $g': X \rightarrow Y$ ,  $g' \simeq g$  (rel  $(X, A)^{k-2}$ ), with  $f|_{(X, A)^{k+1}} = g'|_{(X, A)^{k+1}}$ , if and only if  $d^{k+1}(f, g; A)$  is in the image of  $\Delta^{k-1}(f; A)$ . This is seen by picking  $G_1$ , as above, with  $d^{k+1}(f, G_1; A) = d^{k+1}(g, f; A)$ . By additivity,  $d^{k+1}(g, G_1; A) = 0$ , so

$$g|_{(X, A)^{k-1}} \simeq G_1|_{(X, A)^{k-2}} f.$$

Thus, we call the class of  $d^{k+1}(f, g; A)$  in  $\text{coker}(\Delta^{k-1}(f; A))$  the *secondary obstruction* to deforming  $f$  to  $g$  (rel  $A$ ).

If  $X$  is a 3-manifold and  $k = 2$ , the secondary obstruction is the complete obstruction to finding a homotopy, since it allows us to modify the (constant) homotopy from  $f|_{(X, A)^2}$  to  $g|_{(X, A)^2}$  over all  $(X, A)^2$  before trying to extend to  $(X, A)^3$ .

We will use this obstruction theory in the following situation:  $X = Y = M_0 = (\Sigma/\pi)_0$ ,  $A = \partial M_0$ ,  $f = \text{id}_{M_0}$ ,  $g = \rho$ ,  $\theta = \text{id}$ . Since  $\rho$  has support in a neighborhood of  $\partial M_0$ , it is clear that  $\text{id}|_{(M_0, \partial M_0)^2} = \rho|_{(M_0, \partial M_0)^2}$ . Thus, we have a primary obstruction

$$d^3(\text{id}, \rho; \partial M_0) \in H^3(M_0, \partial; \pi_3(M_0)),$$

and a secondary obstruction given by

$$\Delta^1(\text{id}; \partial M_0) \equiv \Delta: H^1(M_0, \partial; \pi_2(M_0)) \rightarrow H^3(M_0, \partial; \pi_3(M_0)).$$

Notice the correspondence between the descriptions of  $\Delta$  and of the maps on  $\pi_2$  from §4. Given  $[\zeta] \in H^1(M_0, \partial; \mathbb{Z}\pi/N)$ , we build  $\zeta$  into the homotopy  $G$  as follows: Take a relative cell decomposition for  $(M_0, \partial)$  with one 0-cell on  $\partial M_0$  so that 1-cells

$\sigma$  represent elements of  $\pi$ . Define  $G$  to “wrap” the rectangle  $\sigma \times [0, 1]$  around the element  $\zeta(\sigma)$  of  $\pi_2$ . If we regard  $G$  as a partial map  $M_0 \times S^1 \rightarrow M_0 \times_\rho S^1$ , then, as in Proposition 4.3,  $G|_{(M_0 \times S^1)^3}$  determines a map  $\Theta([\zeta]): I\pi \rightarrow \mathbf{Z}\pi/N$ .

Conversely, given  $\phi: I\pi \rightarrow \mathbf{Z}\pi/N$ , define a partial homotopy as follows: If a 1-cell  $\sigma$  represents  $g_\sigma \in \pi$ , define  $G: \sigma \times [0, 1] \rightarrow M_0$  so that  $G_0 = G_1 = \text{id}_\sigma$ , and so that  $G(\sigma \times [0, 1])$  represents  $\phi(g_\sigma - e)$ . On the product of two 1-cells, say  $\sigma\bar{\sigma}$ ,  $G$  represents  $\phi(g_\sigma - e) + g_\sigma\phi(g_{\bar{\sigma}} - e) = \phi(g_\sigma g_{\bar{\sigma}} - e)$ , so  $G$  extends to a homotopy on the 2-cells with  $G_1|_{(M_0, \partial)^2} = \text{id}|_{(M_0, \partial)^2}$ . Extend  $G$  over the top 3-cell, and take  $d^3(\text{id}, G_1; \partial)$ . This is clearly the same procedure used to define  $\Delta$ . If we define  $\Psi(\phi) \in H^1(M_0, \partial; \mathbf{Z}\pi/N)$  by  $\Psi(\phi)(\sigma) = \phi(g_\sigma - e)$ , and define  $\bar{\Delta}: \text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N) \rightarrow H^3(M_0, \partial; \pi_3)$  by  $\bar{\Delta}(\phi) = d^3(\text{id}, G_1; \partial)$ , we have a commutative triangle:

$$(11) \quad \begin{array}{ccc} \text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N) & \xrightleftharpoons[\Theta]{\Psi} & H^1(M_0, \partial; \mathbf{Z}\pi/N) \\ \bar{\Delta} \searrow & & \swarrow \Delta \\ & H^3(M_0, \partial; \pi_3) & \end{array}$$

In fact,  $\psi$  fits nicely into the exact sequences:

$$(12) \quad \begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}_{\mathbf{Z}\pi}(\mathbf{Z}\pi, \mathbf{Z}\pi/N) & \xrightarrow{i^*} & \text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N) & \rightarrow & \text{Ext}_{\mathbf{Z}\pi}^1(\mathbf{Z}, \mathbf{Z}\pi/N) \rightarrow 0 \\ & & \cong \downarrow \Psi & & \cong \downarrow \Psi & & \Psi \downarrow \cong H^2(\pi) \\ 0 & \rightarrow & \cong H^1(M_0, \partial; \mathbf{Z}\pi) & \xrightarrow{\iota} & H^1(M_0, \partial; \mathbf{Z}\pi/N) & \rightarrow & H^2(M_0, \partial; \mathbf{Z}) \cong \rightarrow 0 \end{array}$$

From

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(M_0; \mathbf{Z}\pi) & \rightarrow & H^0(\partial M_0; \mathbf{Z}\pi) & \rightarrow & H^1(M_0, \partial; \mathbf{Z}\pi) \rightarrow 0 \\ & & \parallel & & \parallel & & \\ & & \mathbf{Z}(N) & & \mathbf{Z}\pi & & \end{array}$$

we see that  $[\zeta] \in H^1(M_0, \partial; \mathbf{Z}\pi)$  is given by  $\zeta(\sigma) = (g_\sigma - e)a_\zeta$ ,  $a_\zeta \in \mathbf{Z}\pi$ , where  $[\zeta] = [\zeta'] \Leftrightarrow a_\zeta \equiv a_{\zeta'} \pmod{N}$ . Via  $\Psi$ , such cocycles correspond to maps  $I\pi \rightarrow \mathbf{Z}\pi/N$  which are the restrictions of maps  $\mathbf{Z}\pi \rightarrow \mathbf{Z}\pi/N$ .

We now describe  $\pi_3(M_0)$ . Since  $\tilde{M}_0 \simeq \bigvee_{i=1}^{n-1} S_i^2$ ,  $\pi_3$  is given by a theorem of Hilton [11; 10, p. 150]:

**THEOREM.**  $\pi_3(\bigvee_{i=1}^{n-1} S_i^2) \cong \bigoplus_{i=1}^{n-1} \pi_3(S_i^2) \oplus \bigoplus_{i < j} \pi_3(S_{i,j}^3)$ , where the generator of the infinite cyclic group  $\pi_3(S_i^2)$  ( $\pi_3(S_{i,j}^3)$ ) corresponds to the Hopf map for  $S_i^2$  (Whitehead product  $[S_i^2, S_j^2]$ ).  $\square$

Hopf maps and Whitehead products satisfy the following:

- (1)  $[\ , \ ]: \pi_2 \times \pi_2 \rightarrow \pi_3$  is  $\mathbf{Z}$ -bilinear, symmetric,
- (2)  $\text{Hopf}(x + y) = \text{Hopf}(x) + \text{Hopf}(y) + [x, y]$ ,
- (3)  $[x, x] = 2 \text{Hopf}(x)$ ,
- (4)  $g[x, y] = [gx, gy]$ ,  $\text{Hopf}(gx) = g \text{Hopf}(x)$ ,  $g \in \pi_1$ .

Therefore, if we let  $S^2\pi_2$  denote the symmetric 2-tensors of  $\pi_2$  ( $S^2\pi_2 = \pi_2 \otimes \pi_2 / (x \otimes y - y \otimes x)$ ), we have the following exact sequence of  $\mathbf{Z}\pi$ -modules [10, p. 150]:

$$0 \rightarrow S^2\pi_2 \xrightarrow{\text{Wh}} \pi_3 \xrightarrow{v} \mathbf{Z}_2 \otimes \pi_2 \rightarrow 0,$$

where  $v(\text{Hopf}(x)) = 1 \otimes x$ . This coefficient sequence yields, via duality:

$$\begin{array}{ccccccc} \rightarrow & H^3(M_0, \partial; S^2\pi_2) & \rightarrow & H^3(M_0, \partial; \pi_3) & \xrightarrow{v} & H^3(M_0, \partial; \mathbf{Z}_2 \otimes \pi_2) & \rightarrow 0 \\ & \parallel D & & \parallel D & & \parallel D & \\ (13) \rightarrow & H_0(M_0; S^2\pi_2) & \rightarrow & H_0(M_0; \pi_3) & \rightarrow & H_0(M_0; \mathbf{Z}_2 \otimes \pi_2) & \rightarrow 0 \\ & \parallel & & \parallel & & \parallel & \\ \rightarrow & \mathbf{Z} \otimes_{\pi} S^2\pi_2 & \xrightarrow{1 \otimes \text{Wh}} & \mathbf{Z} \otimes_{\pi} \pi_3 & \xrightarrow{1 \otimes v} & \mathbf{Z}_2 \otimes_{\pi} \pi_2 & \rightarrow 0 \end{array}$$

With notation thus established, we now state the main result of this section:

**THEOREM 7.4** [10, p. 189]. *Let  $M$  be a closed 3-manifold. Then the rotation  $\rho$  in  $\partial M_0$  is homotopic to  $\text{id}_{M_0} (\text{rel } \partial)$  if and only if every summand of  $M$  is either  $S^2 \times S^1$ ,  $S^2 \tilde{\times} S^1$ ,  $\cong P^2 \times S^1$ , or  $\Sigma/\pi$ , where all Sylow subgroups of  $\pi$  are cyclic.*

**PROOF.** The only if direction is Corollary 7.2. For the converse, Lemma 7.3 allows us to argue on the prime summands of  $M$ . The discussion in §2, Cases 2, 3, can be easily modified to show  $\rho \cong \text{id}_{M_0} (\text{rel } \partial)$  if  $M$  is  $S^2 \times S^1$ ,  $S^2 \tilde{\times} S^1$ ,  $P^2 \times S^1$ , or a lens space (since such  $M$  admit  $S^1$ -actions with fixed points). Hence, we assume  $M$  is a topological spherical space form  $\Sigma/\pi$ , but not a lens space.

If  $\pi$  is cyclic, then  $M$  is homotopy equivalent to a lens space  $L = L(n, m)$ . This is because the homotopy type of  $M$  is determined by its fundamental group and its  $k$ -invariant, and all possible  $k$ -invariants (units in  $H^4(\mathbf{Z}_n; \mathbf{Z}) \cong \mathbf{Z}_n$ ) are accounted for by lens spaces (see [3, 17, 24]). Thus,  $(M_0, \partial) \cong (L_0, \partial)$ , and the result follows.

If  $\pi$  is not cyclic, then  $\pi$  is the fundamental group of a prism manifold, as described at the end of §4. Note that  $\pi$  is metacyclic, and is given by a split extension

$$1 \rightarrow \mathbf{Z}_{\alpha} \times \mathbf{Z}_{\beta} \rightarrow \pi \xrightarrow{\hookrightarrow} \mathbf{Z}_{2^k} \rightarrow 1,$$

where  $\alpha$  and  $\beta$  are odd,  $(\alpha, \beta) = 1$ ,  $k \geq 2$ . The generator of  $\mathbf{Z}_{2^k}$  acts by  $-1$  on  $\mathbf{Z}_{\alpha}$  to give  $D_{4\alpha}^*$  or  $D_{2^k\alpha}'$ , and  $\mathbf{Z}_{2^k}$  acts trivially on  $\mathbf{Z}_{\beta}$ . Via abelianization,  $\pi$  maps onto  $\mathbf{Z}_{\beta} \times \mathbf{Z}_{2^k}$ .

Of course, the only known examples are prism manifolds, but this is irrelevant to the proof, since we pass to the cover determined by the 2-Sylow subgroup, and by the above discussion it is not necessary that this be a real lens space.

According to Hendriks [10, p. 172],  $d^3(\text{id}, \rho; \partial M_0)$  is represented by the cocycle whose value on the top 3-cell is  $\text{Hopf}(\partial M_0)$ , the natural generator of  $\pi_3(\partial M_0)$ . This class is mapped by  $(1 \otimes v)D$  to the nontrivial element of  $\mathbf{Z}_2 \otimes_{\pi} \pi_2 = \mathbf{Z}_2 \otimes_{\pi} \mathbf{Z}\pi/N \cong \mathbf{Z}_{(2,n)} = \mathbf{Z}_2$ , since  $n$  is even.

To study  $\Delta$ , Hendriks calculates  $\iota_1$  (see (12)). These correspond to homotopies with support in a neighborhood of  $\partial M_0$ , so the calculation is “universal”. He shows [10, Corollaire, p. 162, and Proposition 1, p. 167] that, given  $\phi \in \mathbf{Z}\pi/N \cong H^1(M_0, \partial; \mathbf{Z}\pi)$ ,  $\Delta\iota(\phi) \in H^3(M_0, \partial; \pi_3)$  takes the value  $[\partial M_0, \phi]$  on the 3-cell of



$(M_0, \partial)$ . In other words, if we view  $\Delta$ , via (11)–(13) as  $D\bar{\Delta}i^*: \text{Hom}_{\mathbf{Z}\pi}(\mathbf{Z}\pi, \mathbf{Z}\pi/N) \rightarrow \mathbf{Z} \otimes_{\pi} \pi_3$ , then  $D\bar{\Delta}i^*(\phi) = 1 \otimes [e, \phi]$ . It is easy to see that the elements  $1 \otimes [e, \phi]$ ,  $\phi \in \pi_2$ , generate  $\mathbf{Z} \otimes_{\pi} S^2\pi_2$ . Thus, we have an induced map

$$\begin{array}{ccc} \text{coker}(\iota) & \rightarrow & \text{coker}(1 \otimes \text{Wh}) \\ \parallel & & \parallel \\ H^2(\pi; \mathbf{Z}) & \xrightarrow{\Phi} & \mathbf{Z}_2 \end{array}$$

and we need to show  $\Phi$  is onto.

Until now, we have given Hendriks' proof. At this point our argument diverges from his, and is motivated by the construction of special isometries in §4. Recall from Lemma 4.5 that a special isometry is constructed by lifting  $\psi: \pi \rightarrow \mathbf{Z}[1/n]/\mathbf{Z}$ ,  $\psi(g^*) = \frac{1}{2}$ , to  $\tilde{\psi}(g) \in \mathbf{Z}[1/n]$ , and choosing  $m_g = \tilde{\psi}(g) - (n-1)/2n$ , with  $m_g + m_{g^{-1}} = 1/n$ . Thus, for  $\pi' \subseteq \pi$ ,  $\tilde{\psi}|_{\pi'}$  will produce a special isometry of  $\pi'$ . In terms of  $\text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N)$ , this means that we first restrict  $I\pi$  to  $I\pi'$ , and then project  $p: \mathbf{Z}\pi/N \rightarrow \mathbf{Z}\pi'/N'$ , by forgetting the elements in  $\pi \setminus \pi'$ . (See proof of Theorem 4.4. Notice that the  $m_g$  change, since  $n$  is replaced by  $|\pi'|$ , but the coefficients of (8) do not change.)

The effect of this procedure on the right-hand side of (12), and hence on  $\Phi$ , is given by:

$$\begin{array}{ccccccc} \phi & \in & \text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N) & \rightarrow & H^1(\pi; \mathbf{Z}\pi/N) & \cong & H^2(\pi; \mathbf{Z}) \\ & & i^* \downarrow & & i^* \downarrow & & i^* \downarrow \\ & & \text{Hom}_{\mathbf{Z}\pi'}(I\pi', \mathbf{Z}\pi/N) & \rightarrow & H^1(\pi'; \mathbf{Z}\pi/N) & \cong & H^2(\pi'; \mathbf{Z}) \\ & & p_* \downarrow & & p_* \downarrow \cong & & \parallel \\ \phi' & \in & \text{Hom}_{\mathbf{Z}\pi'}(I\pi', \mathbf{Z}\pi'/N') & \rightarrow & H^1(\pi'; \mathbf{Z}\pi'/N') & \cong & H^2(\pi'; \mathbf{Z}) \end{array}$$

The isomorphism  $p_*: H^1(\pi'; \mathbf{Z}\pi/N) \rightarrow H^1(\pi'; \mathbf{Z}\pi'/N')$  arises in the following way: As a left  $\mathbf{Z}\pi'$ -module,  $\mathbf{Z}\pi/N \cong (\oplus \mathbf{Z}\pi')/N$ , the sum being taken over the cosets  $\pi' \setminus \pi$ . Also,  $p(N) = N'$ . Hence we find:

$$\begin{array}{ccccccc} 0 = \oplus H^1(\pi'; \mathbf{Z}\pi') & \rightarrow & H^1(\pi'; \mathbf{Z}\pi/N) & \rightarrow & H^2(\pi'; \mathbf{Z}) & \rightarrow & \oplus H^2(\pi'; \mathbf{Z}\pi') = 0 \\ & & p_* \downarrow & & \parallel & & \\ & & H^1(\pi'; \mathbf{Z}\pi'/N') & \rightarrow & H^2(\pi'; \mathbf{Z}) & & \end{array}$$

Thus, the process of passing from  $\phi$  to  $\phi'$ , measured on  $H^2$ , is just given by  $i^*: H^2(\pi) \rightarrow H^2(\pi')$ .

This has a nice geometric interpretation. Let  $\bar{M} \xrightarrow{q} M_0$  be the cover of  $M_0$  corresponding to  $\pi'$ , with a lift  $*$  of  $* \in \partial M_0$ . Note that  $\bar{M}$  has  $[\pi: \pi']$  boundary 2-spheres, each one a natural generator of a  $\mathbf{Z}\pi'$  component in  $\pi_2(\bar{M}) \cong \mathbf{Z}\pi/N \cong (\oplus \mathbf{Z}\pi')/N$ . Now  $\phi \in \text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N)$  determines a homotopy on the 1-cells of  $M_0$ . The lift of this homotopy corresponds to  $i^*(\phi)$ . Add  $[\pi: \pi'] - 1$  copies of  $D^3$  to  $\bar{M}$ , capping off all boundary components except the one containing  $*$ , and let  $M'$



Since  $\bar{\Gamma}$  is induced by a map of odd degree,  $\bar{\Gamma} = 1$ . Hence, we find:

$$\begin{array}{ccc} H^2(\pi; \mathbf{Z}) & \xrightarrow{\Phi} & \mathbf{Z}_2 \\ i^* \downarrow & & \parallel \\ H^2(\pi'; \mathbf{Z}) & \xrightarrow{\Phi'} & \mathbf{Z}_2 \end{array}$$

Since  $\rho \simeq \text{id}_{M'} \text{ (rel } \partial)$ ,  $\Phi'$  is onto. Also, since  $i: \pi' \hookrightarrow \pi$  splits,  $i^*$  is onto. Thus we conclude  $\Phi$  is onto, and the proof of Theorem 7.4 is complete.  $\square$

At this point, the proof is not completely satisfying, since it merely shows that  $\Delta([\zeta]) = d^3(\text{id}, \rho; \partial M_0)$  for some  $[\zeta] \in H^1(M_0, \partial; \mathbf{Z}\pi/N)$ , but we have no explicit  $\zeta$ . Via  $\theta$ ,  $[\zeta]$  corresponds to  $-\phi \in \text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N)$ , where we know from §4 that  $\phi$  is a special isometry. Understanding which  $-\phi$  satisfy  $\bar{\Delta}(-\phi) = d^3(\text{id}, \rho; \partial M_0)$  is the same as understanding what the homotopy from  $\text{id}_{M_0}$  to  $\rho \text{ (rel } \partial)$  looks like on the 1-cells of  $(M_0, \partial M_0)$ .

Recall that a special isometry begins with  $\psi: \pi \rightarrow \mathbf{Z}[1/n]/\mathbf{Z}$ , such that  $\psi(g^*) = \frac{1}{2}$ . If  $\pi$  is even order cyclic, write  $\pi = \mathbf{Z}_{2^k\beta}$ ,  $\beta$  odd, generated by  $g$ . Then the map  $\psi(g^i) = ij/2^k\beta$  will satisfy  $\psi(g^*) = \frac{1}{2}$  provided  $j$  is odd. If  $\pi$  is noncyclic, then  $H_1(\pi) = \mathbf{Z}_{2^k\beta}$ . Since  $\psi$  must factor through  $H_1(\pi)$ , the result is the same. So in either case there are  $2^{k-1}\beta$  possible  $\psi$ 's. Our first result is that  $D\bar{\Delta}$  is constant on special isometries arising from a given  $\psi$ .

**PROPOSITION 7.5.** *Let  $\phi_1, \phi_2 \in \text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N)$  be special isometries arising from two lifts of  $\psi: \pi \rightarrow \mathbf{Z}[1/n]/\mathbf{Z}$ . Then  $D\bar{\Delta}(\phi_1) = D\bar{\Delta}(\phi_2)$ .*

**PROOF.** From (8),  $\phi_1 - \phi_2$  is given by

$$h - e \mapsto - \sum_{g \neq e, h} \left\{ (m_g^1 - m_g^2) + (m_{h^{-1}}^1 - m_{h^{-1}}^2) + (m_{h^{-1}g}^1 - m_{h^{-1}g}^2) \right\} g,$$

where  $m_g^i \in \mathbf{Z}[1/n]$  is given by  $\tilde{\psi}^i(g) - (n-1)/2n$ ,  $\tilde{\psi}^i$  is a lift of  $\psi$ , and  $m_g^i + m_{g^{-1}}^i = 1/n$ ,  $i = 1, 2$ . Note that  $m_g^1 - m_g^2 \in \mathbf{Z}$  and  $m_g^1 - m_g^2 = -(m_{g^{-1}}^1 - m_{g^{-1}}^2)$ .

Now, it is easy to compute that if  $\phi \in \text{Hom}_{\mathbf{Z}\pi}(\mathbf{Z}\pi, \mathbf{Z}\pi/N)$  is given by  $\phi(e) = \sum_{g \neq e} n_g g$ ,  $n_g \in \mathbf{Z}$ , then its restriction  $i^*(\phi)$  is given by

$$h - e \mapsto - \left[ \sum_{g \neq e, h} (n_g + n_{h^{-1}} - n_{h^{-1}g})g + (n_h + n_{h^{-1}})h \right].$$

Letting  $n_g = m_g^1 - m_g^2$ , we have  $\phi_1 - \phi_2 = i^*(\phi)$ , and  $n_g = -n_{g^{-1}}$ . Hence

$$D\bar{\Delta}(\phi_1 - \phi_2) = D\bar{\Delta}i^*(\phi) = 1 \otimes [e, \phi(e)] = 1 \otimes \sum_{g \neq e} n_g (e \otimes g) = 0,$$

since  $e \otimes g = e \otimes g^{-1}$ .  $\square$

We interpret this as follows. Consider the diagram:

$$\begin{array}{ccc}
 0 & & 0 \\
 \uparrow & & \uparrow \\
 H^2(\pi) & \xrightarrow{\phi} & \mathbf{Z}_2 \\
 \uparrow & & \uparrow \\
 \text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N) & \xrightarrow{D\bar{\Delta}} & \mathbf{Z} \otimes_{\pi} \pi_3 \\
 \uparrow i^* & & \uparrow 1 \otimes \text{Wh} \\
 \text{Hom}_{\mathbf{Z}\pi}(\mathbf{Z}\pi, \mathbf{Z}\pi/N) & \xrightarrow{D\bar{\Delta}i^*} & \mathbf{Z} \otimes_{\pi} S^2\pi_2 \\
 \uparrow & & \\
 0 & & 
 \end{array}
 \quad (14)$$

Clearly  $\psi$  determines the class in  $H^2(\pi)$  of  $\phi_1, \phi_2$ , so  $\phi_1 - \phi_2 = i^*(\phi)$ . Since  $D\bar{\Delta}i^*(\phi) = 0$ , we see that the special isometries associated to a given  $\psi$  lie in an affine subspace of  $\text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N)$ .

Now look at the terms on the right. Up to elements of  $\pi$ , every generator of  $S^2\pi_2$  can be written as  $e \otimes g = g \otimes e = e \otimes g^{-1}$ , and we see that

$$\mathbf{Z} \otimes_{\pi} S^2\pi_2 \cong \mathbf{Z}\pi / (N, g - g^{-1}),$$

a free abelian group rank  $n/2$ , and  $D\bar{\Delta}i^*$  corresponds to the natural map  $\mathbf{Z}\pi/N \rightarrow \mathbf{Z}\pi/(N, g - g^{-1})$ , with rank (kernel) =  $n/2 - 1$ .

The generators of  $\pi_3$  are either  $\text{Hopf}(g)$  or  $[g, h] = [h, g]$ ,  $g \neq h$ . Up to elements of  $\pi$ ,  $[g, h] = [e, g^{-1}h]$ . Furthermore,

$$[e, g] = \left[ - \sum_{h \neq e} h, g \right] = -[g, g] - \sum_{h \neq e, g} [h, g] = -2 \text{Hopf}(g) - \sum_{h \neq e, g} [e, h],$$

so that  $2 \text{Hopf}(g) = \sum_{h \neq e} [e, h]$ . Thus, the terms  $[g, h]$  contribute terms  $[e, g] = [e, g^{-1}]$ ,  $g \neq e$ , the terms  $\text{Hopf}(g)$  contribute another  $\mathbf{Z}$  factor, generated, say, by  $a$ , and we see that

$$\mathbf{Z} \otimes_{\pi} \pi_3 \cong (\mathbf{Z}(\pi - e)/(g - g^{-1}) \oplus \mathbf{Z}(a)) / \left( 2a + \sum_{g \neq e} g \right).$$

(Note that we have another relation given by

$$\begin{aligned}
 g^{-1} \text{Hopf}(g) &= \text{Hopf}(e) = \text{Hopf}\left(- \sum_{h \neq e} h\right) = \text{Hopf}\left(\sum_{h \neq e} h\right) \\
 &= \sum_{i=1}^{n-1} \text{Hopf}(g_i) + \sum_{i < j} [g_i, g_j],
 \end{aligned}$$

using  $\text{Hopf}(x + y) = \text{Hopf}(x) + \text{Hopf}(y) + [x, y]$ . But it is easy to see this gives a multiple of the previous relation.)

Order the elements of  $\pi$  so that  $e = g_0$ ,  $g_i^{-1} = g_{n-i}$ ,  $g_{n/2} = g^*$ , the unique element of order 2. We now have

$$\begin{array}{ccc} \mathbf{Z} \otimes_{\pi} \pi_3 & = & \mathbf{Z}(a, g_1, \dots, g_{n/2}) / (2a + 2g_1 + \dots + 2g_{n/2-1} + g_{n/2}) \\ \uparrow 1 \otimes \text{Wh} & & \uparrow \begin{array}{cc} 2a & g_i \\ \uparrow & \uparrow \\ g_0 & g_i \end{array} \\ \mathbf{Z} \otimes_{\pi} S^2 \pi_2 & = & \mathbf{Z}(g_0, \dots, g_{n/2}) / (g_0 + 2g_1 + \dots + 2g_{n/2-1} + g_{n/2}), \end{array}$$

so  $1 \otimes \text{Wh}$  is an injection, with  $\text{coker}(1 \otimes \text{Wh}) = \mathbf{Z}_2(a)$ . Of course,  $a = D(d^3(\text{id}, \rho; \partial M_0))$ .

We now know that both  $D\bar{\Delta}$  and  $D\bar{\Delta}i^*$  are surjective, with kernels of rank  $n/2 - 1$ . The kernel of  $D\bar{\Delta}i^*$  is easily understood, as in the proof of Proposition 7.5. The map  $D\bar{\Delta}$  is not so well understood, but at least its kernel has rank  $n/2 - 1$ , large enough to accommodate the affine subspaces of special isometries from Proposition 7.5. This proposition was the first step in deciding which special isometries  $\phi$  satisfy  $D\bar{\Delta}(-\phi) = a$ . In fact, we now prove that *all* special isometries have this property:

**THEOREM 7.6.** *Let  $\phi$  be a special isometry. Then  $D\bar{\Delta}(-\phi) = a$ . In other words, there is a homotopy  $\text{id}_{M_0} \simeq \rho \text{ (rel } \partial)$  which represents  $-\phi$  on the 1-cells of  $(M_0, \partial M_0)$ .*

**PROOF.** First consider the cyclic case  $\pi = \mathbf{Z}_{2^k \beta} = \mathbf{Z}_n$ . We can readily understand (14) via polynomials, as in the example of §4, and we find:

$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & \xrightarrow{\phi} & \uparrow \\ \mathbf{Z}_n & & \mathbf{Z}_2 \\ \uparrow & & \uparrow \\ \frac{\mathbf{Z}[x]}{\begin{pmatrix} x^n - 1 \\ 1 + x + \dots + x^{n-1} \end{pmatrix}} & \xrightarrow{D\bar{\Delta}} & \frac{\mathbf{Z}(a) \oplus \{a_1x + \dots + a_{n/2}x^{n/2}\}}{(2a + 2x + \dots + 2x^{n/2-1} + x^{n/2})} \\ \uparrow i^* & & \uparrow \text{Wh} \\ \frac{\mathbf{Z}[x]}{\begin{pmatrix} x^n - 1 \\ 1 + x + \dots + x^{n-1} \end{pmatrix}} & \xrightarrow{D\bar{\Delta}i^*} & \frac{\{a_0 + a_1x + \dots + a_{n/2}x^{n/2}\}}{(1 + 2x + \dots + 2x^{n/2-1} + x^{n/2})} \\ \uparrow & & \uparrow \\ 0 & & 0 \end{array}$$

From §4,  $i^*(q(x)) = (x - 1)q(x)$ , and  $\phi \in \text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N)$  is determined by  $\phi(g - e) = p(x)$ . Note that the value of  $p(x)$  in  $\mathbf{Z}_n$  is given by its augmentation  $p(1)$ .

Now, we know from the geometry of lens space (using the  $S^1$ -action) that there is a homotopy  $\text{id}_{L_0} \simeq \rho \text{ (rel } \partial)$ , inducing  $X \in \text{Hom}_{\mathbf{Z}\pi}(I\pi, \mathbf{Z}\pi/N)$ . The special isometry  $-X$  is determined by  $-X(g - e) = -x$  (see §4), so we have  $D\bar{\Delta}(x) = a$ .

In Remark (2) following Lemma 4.5, we claimed that  $-X$  corresponds to  $\mathbf{Z}_n \rightarrow \mathbf{Z}[1/n]/\mathbf{Z}$ ,  $g \mapsto -1/n$ . More generally, consider  $\psi_j: \mathbf{Z}_n \rightarrow \mathbf{Z}[1/n]/\mathbf{Z}$  given by  $\psi_j(g^i) = ij/n$ ,  $j$  odd. It is easy to see that if we pick lifts  $\tilde{\psi}_j(g^i) = ij/n - (j-1)/2$ , then  $m_{g^i} = \tilde{\psi}_j(g^i) - (n-1)/2n = ij/n + 1/2n - j/2$  will give a special isometry, say  $-X_j$ . The  $(g^i)$ th coefficient,  $i > 1$ , of  $-X_j(g-e)$  is given, from (8), by  $m_{g^i} + m_{g^{n-1}} - m_{g^{i-1}} + (n-1)/2n = (j+1)/2$ . This means that the polynomial associated to  $X_j$  is

$$p_j(x) = x + \frac{j+1}{2}(x^2 + \cdots + x^{n-1}).$$

To compute  $D\bar{\Delta}(p_j(x))$ , observe  $p_j(1) = 1 + (j+1)(n-2)/2$ , so we can write

$$\begin{aligned} p_j(x) - p_j(1) &= \frac{-(j+1)(n-2)}{2}x + \frac{j+1}{2}(x^2 + \cdots + x^{n-1}) \\ &= (x-1)q_j(x). \end{aligned}$$

To find  $q_j(x)$ , we calculate

$$\begin{aligned} &-\frac{j+1}{2}[(n-2)x - (x^2 + \cdots + x^{n-1})] \\ &= -\frac{j+1}{2}[(1-x) + nx] \\ &= -\frac{j+1}{2}[(1-x) - (1-x)(x^2 + 2x^3 + 3x^4 + \cdots + (n-1)x^n)] \\ &= -\frac{j+1}{2}(x-1)(n-2 + x^2 + 2x^3 + \cdots + (n-2)x^{n-1}) \\ &\quad \text{mod} \begin{pmatrix} x^n - 1 \\ 1 + x + \cdots + x^{n-1} \end{pmatrix}. \end{aligned}$$

Hence, we find

$$\begin{aligned} D\bar{\Delta}(p_j(x)) &= D\bar{\Delta}(p_j(x) - p_j(1)x) + D\bar{\Delta}(p_j(1)x) \\ &= D\bar{\Delta}i^* \left[ -\frac{j+1}{2}(n-2 + x^2 + 2x^3 + \cdots + (n-2)x^{n-1}) \right] \\ &\quad + a \left( 1 + \frac{(j+1)(n-2)}{2} \right) \\ &= -\frac{j+1}{2} \text{Wh} \left[ (n-2) \left( 1 + x + \cdots + x^{n/2-1} + \frac{1}{2}x^{n/2} \right) \right] \\ &\quad + a \left( 1 + \frac{(j+1)(n-2)}{2} \right) \\ &= -\frac{(j+1)(n-2)}{2} \left( 2a + x + \cdots + x^{n/2-1} + \frac{1}{2}x^{n/2} \right) \\ &\quad + a \left( 1 + \frac{(j+1)(n-2)}{2} \right) \\ &= -\frac{(j+1)(n-2)}{4}(2a) + a \left( 1 + \frac{(j+1)(n-2)}{2} \right) = a. \end{aligned}$$

This computation, together with Proposition 7.5, proves the theorem in the cyclic case.

In fact, there is a simple geometric explanation for the above calculation, arising from the “belt trick,” reflecting the kernel of  $\pi_1(SO(2)) \rightarrow \pi_1(SO(3))$ . Given  $j$  odd, let  $S^1$  act on the lens space by  $(-j)$  times the original action. This provides a homotopy  $\text{id}_{L_0} \simeq \rho^{-j} \text{ (rel } \partial)$ , corresponding to the polynomial  $-jx$ . But  $\rho^2 \simeq \text{id}_{L_0}$ . This homotopy is supported in a collar of  $\partial L_0$ , and alters arcs emanating from  $\partial L_0$  by the belt trick, i.e., wraps the arcs once around  $\partial L_0$ . Then the closed curve  $g$  (Figure 1) will be wrapped around  $e - g$ , corresponding to the polynomial  $-2x - x^2 - \dots - x^{n-1}$ . Thus, following our homotopy  $\text{id}_{L_0} \simeq \rho^{-j} \text{ (rel } \partial)$  by  $-(1+j)/2$  belt tricks, we find a homotopy  $\text{id}_{L_0} \simeq \rho \text{ (rel } \partial)$  with polynomial  $p_j(x)$ , so that  $D\bar{\Delta}(p_j(x)) = a$ .

Note that we started with  $x$ , multiplied it to  $-jx$ , and then “corrected” it by the belt trick to  $p_j(x)$ . Thus, we see that once a special isometry, corresponding to *one*  $\psi: \pi \rightarrow \mathbb{Z}[1/n]/\mathbb{Z}$ , provides a homotopy  $\text{id}_{L_0} \simeq \rho$ , then the special isometries corresponding to *all*  $\psi$  also provide homotopies.

This observation now gives the proof when  $\pi$  is noncyclic. If a special isometry  $\phi \in \text{Hom}_{\mathbb{Z}\pi}(I\pi, \mathbb{Z}\pi/N)$ , corresponding to one  $\psi: \pi \rightarrow \mathbb{Z}[1/n]/\mathbb{Z}$ , provides a homotopy  $\text{id}_{M_0} \simeq \rho \text{ (rel } \partial)$  (by Theorem 7.4,  $\phi$  exists), then  $-j\phi$ , corrected by  $-(1+j)/2$  belt tricks, also gives a homotopy. Since the belt trick arises from  $\text{Hom}_{\mathbb{Z}\pi}(\mathbb{Z}\pi, \mathbb{Z}\pi/N)$ , it is not detected in  $H^2(\pi)$ , and we see that special isometries corresponding to all odd multiples of the generator of  $H^2(\pi)$  provide homotopies. This finally proves Theorem 7.6.  $\square$

EXAMPLE. The smallest noncyclic group here is  $D_{12}^* = (A, B \mid A^3 = B^4 = 1, BAB^{-1} = A^{-1})$ , where  $H_1(D_{12}^*) \cong \mathbb{Z}_4(B)$ . Let  $\psi: D_{12}^* \rightarrow \mathbb{Z}[\frac{1}{4}]/\mathbb{Z}$  be given by  $\psi(A^i B^j) = -j/4$ ,  $j = 0, 1, 2, 3$ . Pick lifts  $\tilde{\psi}(A^i B^j)$  as follows:

$g$	$B$	$B^2$	$B^3$	$A$	$AB$	$AB^2$	$AB^3$	$A^2$	$A^2B$	$A^2B^2$	$A^2B^3$
$24m_g$	7	1	-5	13	7	1	-5	-11	7	1	-5

One easily computes that the negative of the special isometry is given by

$$B - e \rightarrow B + A + AB, \quad A - e \rightarrow A - A^2.$$

Note that the restriction/projection procedure of Theorem 7.4 gives the standard map  $(B - e \rightarrow B)$  on  $L(4, 1)$ .

The prism manifold with  $\pi_1 = D_{12}^*$  is the Seifert manifold  $(-1; (2, 1), (2, 1), (3, 1))$ , and can also be described by glueing a solid torus to a twisted  $I$ -bundle over the Klein bottle  $K$  so as to kill the loop  $a^3 b^2$ , where  $a, b^2$  are standard generators of  $T^2$  corresponding to the standard presentation  $(a, b \mid bab^{-1} = a^{-1})$  of  $\pi_1(K)$ .

Prism manifolds are quite well understood. In particular, Rubinstein has computed their mapping class groups [33]. Nevertheless, even for this example it seems rather difficult to “see” the homotopy  $\text{id} \simeq \rho \text{ (rel } \partial)$  whose effect on the 1-cells is given above. For larger groups, it is presumably more difficult. In fact, Friedman and Witt have recently shown why the homotopy is difficult to see:

**THEOREM [6].** *The rotation  $\rho$  is not isotopic to the identity for the prism manifolds  $S^3/D_{2^k\alpha}'$ ,  $S^3/(D_{2^k\alpha}' \times \mathbb{Z}_\beta)$ , and  $S^3/(D_{4\alpha}' \times \mathbb{Z}_\beta)$ .*

Their theorem is based on work of Ivanov [12], which does not include the case  $S^3/D_{4\alpha}^*$ . Using these manifolds in forming connected sums, one can construct (the *only* known examples of) homeomorphisms of closed 3-manifolds which are homotopic but not isotopic to the identity [6].

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DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027 (Current address)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48109